Fast arithmetics in Artin-Schreier towers over finite fields

Luca De Feo[1]
joint work with É. Schost[2]

[1]École Polytechnique and INRIA, France
[2]ORCCA and CSD, The University of Western Ontario, London, ON

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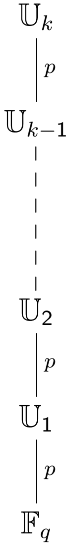


Standard arithmetics

$$+, -, \times, / \ : \begin{cases} \mathbb{U}_i \times \mathbb{U}_i & \rightarrow \ \mathbb{U}_i \\ (u, v) & \mapsto \ u \,\mathsf{op}\, v \end{cases}$$

Doing arithmetics in towers of extensions

$$\begin{array}{c} \mathbb{U}_k \\ \Big|\, p \\ \mathbb{U}_{k-1} \\ \vdots \\ \mathbb{U}_2 \\ \Big|\, p \\ \mathbb{U}_1 \\ \Big|\, p \\ \mathbb{F}_q \end{array}$$

Inclusion

$$\iota \ : \begin{cases} \mathbb{U}_i & \subset \ \mathbb{U}_{i+1} \\ v & \mapsto \ \bar{v} \end{cases}$$

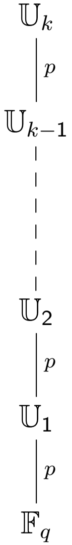
Doing arithmetics in towers of extensions

$$\mathbb{U}_k \overset{p}{\supset} \mathbb{U}_{k-1} \supset \cdots \supset \mathbb{U}_2 \overset{p}{\supset} \mathbb{U}_1 \overset{p}{\supset} \mathbb{F}_q$$

Membership

$$\iota^{-1} : \begin{cases} \mathbb{U}_{i+1} & \supset & \mathbb{U}_i \\ \iota(v) & \mapsto & v \end{cases}$$

Doing arithmetics in towers of extensions



Projection

$$\pi : \begin{cases} \mathbb{U}_{i+1} & \xrightarrow{\sim} \mathbb{U}_i^p \simeq \mathbb{U}_i[\gamma] \\ v & \mapsto (v_0, \ldots, v_{p-1}) \end{cases}$$

$$\pi^{-1} : \begin{cases} \mathbb{U}_i^p \simeq \mathbb{U}_i[\gamma] & \xrightarrow{\sim} \mathbb{U}_{i+1} \\ (v_0, \ldots, v_{p-1}) & \mapsto \sum_j v_j \gamma^j \end{cases}$$

Doing arithmetics in towers of extensions

$\mathbb{U}_k$

$| \; p$

$\mathbb{U}_{k-1}$

$\vdots$

$\mathbb{U}_2$

$| \; p$

$\mathbb{U}_1$

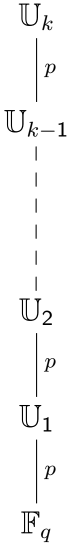
$| \; p$

$\mathbb{F}_q$

Traces

$$\mathsf{Tr} \; : \begin{cases} \mathbb{U}_{i+1} & \longrightarrow \; \mathbb{U}_i \\ v & \longmapsto \; \mathsf{Tr}(v) \end{cases}$$

Doing arithmetics in towers of extensions

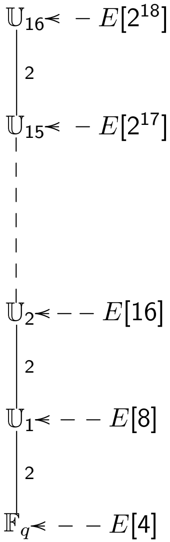


Galois action

$$\varphi \; : \begin{cases} G \times \mathbb{U}_i & \to \; \mathbb{U}_i \\ (\sigma, v) & \mapsto \; \sigma(v) \end{cases}$$

$$G := \mathsf{Gal}(\mathbb{U}_{i+1}/\mathbb{U}_i) \simeq \mathbb{Z}/p\mathbb{Z}$$

Crypto application : Isogeny computation



E, E' elliptic curves
with $\#E(F_q) = \#E'(F_q)$

Theorem/Algorithm

Knowing $E[2^{k+3}]$ and $E'[2^{k+3}]$
$\Rightarrow$ all isogenies of degree $< 2^k$

Example

- $\mathbb{F}_q = \mathbb{F}_{2^{163}}$,
- $E[4] \subset E(\mathbb{F}_q)$, $E[2^{i+2}] \subset E(\mathbb{U}_i)$,
- Isogeny degree $< 2^{15}$ $\Rightarrow$ 16 levels !!
- One element of $\mathbb{U}_{16} \sim 1.5$MB !!

Our context

$$\mathbb{U}_k = \frac{\mathbb{U}_{k-1}[X_k]}{P_{k-1}(X_k)}$$

$$\Big|\, p$$

$$\mathbb{U}_{k-1}$$

$$\vdots$$

$$\mathbb{U}_1 = \frac{\mathbb{U}_0[X_1]}{P_0(X_1)}$$

$$\Big|\, p$$

$$\mathbb{F}_q = \frac{\mathbb{F}_p[X_0]}{Q(X_0)}$$

Tower over finite fields

P_i irreducible polynomial in $\mathbb{U}_i[X]$

Our context

$$\mathbb{U}_k = \frac{\mathbb{U}_{k-1}[X_k]}{P_{k-1}(X_k)}$$

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Tower over finite fields

P_i irreducible polynomial in $\mathbb{U}_i[X]$

But this is too hard.

Artin-Schreier

Definition (Artin-Schreier polynomial)

$\mathbb{K}$ a field of characteristic p, $\alpha \in \mathbb{K}$

$$X^p - X - \alpha$$

is an Artin-Schreier polynomial.

Theorem

$\mathbb{K}$ finite. $X^p - X - \alpha$ irreducible $\Leftrightarrow$ $\mathrm{Tr}_{\mathbb{K}/\mathbb{F}_p}(\alpha) \neq 0$.
If $\eta \in \mathbb{K}$ is a root, then $\eta + 1, \ldots, \eta + (p-1)$ are roots.

Definition (Artin-Schreier extension)

$\mathcal{P}$ an irreducible Artin-Schreier polynomial.

$$\mathbb{L} = \mathbb{K}[X]/\mathcal{P}(X).$$

$\mathbb{L}/\mathbb{K}$ is called an Artin-Schreier extension.

Our context

$$\mathbb{U}_k = \frac{\mathbb{U}_{k-1}[X_k]}{P_{k-1}(X_k)}$$

$$\Big|\, p$$

$$\mathbb{U}_{k-1}$$

$$\vdots$$

$$\mathbb{U}_1 = \frac{\mathbb{U}_0[X_1]}{P_0(X_1)}$$

$$\Big|\, p$$

$$\mathbb{U}_0 = \mathbb{F}_{p^d} = \frac{\mathbb{F}_p[X_0]}{Q(X_0)}$$

Towers over finite fields

$$P_i = X^p - X - \alpha_i$$

We say that $(\mathbb{U}_0, \dots, \mathbb{U}_k)$ is defined by $(\alpha_0, \dots, \alpha_{k-1})$ over $\mathbb{U}_0$.

ANY separable extension of degree p can be expressed this way

Size, complexities

$$\#\mathbb{U}_i = p^{p^i d}$$

$\mathbb{U}_k$

$\mathbb{U}_{k-1}$

$\mathbb{U}_1$

$\mathbb{U}_0$

Optimal representation

All common representations achieve it: $O(p^i d)$

Complexities

optimal:	$O(p^i d)$	addition
quasi-optimal:	$\tilde{O}(i^a p^i d)$	FFT multiplication
almost-optimal:	$\tilde{O}(i^a p^{i+b} d)$	
suboptimal:	$\tilde{O}(i^a p^{i+b} d^c)$	
too bad:	$\tilde{O}\left(i^a (p^{i+b})^e d^c\right)$	naive multiplication

Multiplication function $\mathsf{M}(n)$

FFT: $\mathsf{M}(n) = O(n \log n \log \log n)$, Naive: $\mathsf{M}(n) = O(n^2)$.

Outline

Representation matters!

$\mathbb{U}_k$

$\mathbb{U}_{k-1}$

$\mathbb{U}_1$

$\mathbb{U}_0$

Multivariate representation of $v \in \mathbb{U}_i$

$$v = X_0^{d-1} X_1^{p-1} \cdots X_i^{p-1} + 2X_0^{d-1} X_1^{p-1} \cdots X_i^{p-2} + \cdots$$

Univariate representation of $v \in \mathbb{U}_i$

- $\mathbb{U}_i = \mathbb{F}_p[x_i]$,
- $v = c_0 + c_1 x_i + c_2 x_i^2 + \cdots + c_{p^i d-1} x_i^{p^i d-1}$ with $c_i \in \mathbb{F}_p$.

How much does it cost to...

- Multiply?
- Express the embedding $\mathbb{U}_{i-1} \subset \mathbb{U}_i$?
- Express the vector space isomorphism $\mathbb{U}_i = \mathbb{U}_{i-1}^p$?
- Switch between the representations?

A primitive tower

$\mathbb{U}_k$
$\mathbb{U}_{k-1}$
$\mathbb{U}_1$
$\mathbb{U}_0$

Definition (Primitive tower)

A tower is primitive if $\mathbb{U}_i = \mathbb{F}_p[X_i]$.

In general this is not the case. Think of $P_0 = X^p - X - 1$.

Theorem (extends a result in [Cantor '89])

Let $x_0 = X_0$ *such that* $\mathrm{Tr}_{\mathbb{U}_0/\mathbb{F}_p}(x_0) \neq 0$ *, let*

$$
\begin{aligned}
P_0 &= X^p - X - x_0\\
P_i &= X^p - X - x_i^{2p-1}
\end{aligned}
$$

with x_{i+1} *a root of* P_i *in* $\mathbb{U}_{i+1}$.
Then, the tower defined by $(P_0, \ldots, P_{k-1})$ *is primitive.*

Some tricks to play when $p = 2$.

Computing the minimal polynomials

$\mathbb{U}_k$

$\mathbb{U}_{k-1}$

$\mathbb{U}_1$

$\mathbb{U}_0$

We look for Q_i, the minimal polynomial of x_i over $\mathbb{F}_p$

Algorithm [Cantor '89]

- $Q_0 = Q$ easy,
- $Q_1 = Q_0(X^p - X)$ easy,

Let ω be a $2p-1$-th root of unity,

- $q_{i+1}(X^{2p-1}) = \prod_{j=0}^{2p-2} Q_i(\omega^j X)$ not too hard,
- $Q_{i+1} = q_{i+1}(X^p - X)$ easy.

Complexity

$$O\left(\mathsf{M}(p^{i+2}d)\log p\right)$$

Yes, we can multiply !

$$\begin{array}{c} \mathbb{U}_k \\ \big| \, p \\ \mathbb{U}_{k-1} \\ \vdots \\ \mathbb{U}_2 \\ \big| \, p \\ \mathbb{U}_1 \\ \big| \, p \\ \mathbb{F}_q \end{array}$$

Standard arithmetics

$$+, -, \times, / \ : \begin{cases} \mathbb{U}_i \times \mathbb{U}_i & \rightarrow \ \mathbb{U}_i \\ (u, v) & \mapsto \ u \,\mathsf{op}\, v \end{cases}$$

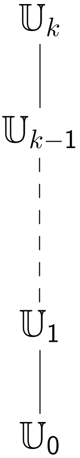
Outline

Level embedding

$$\mathbb{U}_k$$
$$|$$
$$\mathbb{U}_{k-1}$$
$$\vdots$$
$$\mathbb{U}_1$$
$$|$$
$$\mathbb{U}_0$$

$$\pi \quad : \begin{cases} \mathbb{U}_{i+1} & \xrightarrow{\sim} \quad \mathbb{U}_i^p \simeq \mathbb{U}_i[\gamma] \\ v & \mapsto \quad (v_0, \ldots, v_{p-1}) \end{cases}$$

$$\pi^{-1} \quad : \begin{cases} \mathbb{U}_i^p \simeq \mathbb{U}_i[\gamma] & \xrightarrow{\sim} \quad \mathbb{U}_{i+1} \\ (v_0, \ldots, v_{p-1}) & \mapsto \quad \sum_j v_j \gamma^j \end{cases}$$



Push-down

Input $v \dashv \mathbb{U}_i$,
Output $v_0, \ldots, v_{p-1} \dashv \mathbb{U}_{i-1}$ such that $v = v_0 + \cdots + v_{p-1}x_i^{p-1}$.

Lift-up

Input $v_0, \ldots, v_{p-1} \dashv \mathbb{U}_{i-1}$,
Output $v \dashv \mathbb{U}_i$ such that $v = v_0 + \cdots + v_{p-1}x_i^{p-1}$.

Complexity function $\mathsf{L}(i)$

It turns out that the two operations lie in the same complexity class, we note $\mathsf{L}(i)$ for it:

$$\mathsf{L}(i) \;=\; O\left(p\mathsf{M}(p^i d) + p^{i+1} d \log_p (p^i d)^2\right)$$

Push-down

Input $v \dashv \mathbb{U}_i$,
Output $v_0, \ldots, v_{p-1} \dashv \mathbb{U}_{i-1}$ s.t. $v = v_0 + \cdots + v_{p-1}x_i^{p-1}$.

1. Reduce v modulo $x_i^p - x_i - x_{i-1}^{2p-1}$ by a divide-and-conquer approach,
2. each of the coefficients of x_i has degree in x_{i-1} less than $2\deg_{x_i}(v)$,
3. reduce each of the coefficients.

Lift-up

Theorem

Up to some simple formulae:

$$\left(\quad \pi^{-1} \quad \right) \left(v \right) \sim \left(\quad \pi^T \quad \right) \left(\quad M_v^T \quad \right) \left(\mathsf{Tr}^T \right)$$

Transposed algorithms (see [Bürgisser, Clausen and Shokrollahi '97])

- Tr can be easily computed through the *residue formula*.
- *Linear algorithms* can be *transposed* much like linear applications;
- computing $v \cdot \mathsf{Tr} := (M_v)(\mathsf{Tr}^T)$ is *transposed multiplication*.
- Computing π^T is *transposed push-down*.

Lift-up

Theorem

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Theorem

Up to some simple formulae:

$$\begin{pmatrix} & \pi^{-1} & \end{pmatrix} \begin{pmatrix} v \end{pmatrix} \sim \begin{pmatrix} & \pi^T & \end{pmatrix} \begin{pmatrix} & M_v^T & \end{pmatrix} \begin{pmatrix} \mathsf{Tr}^T \end{pmatrix}$$

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- Computing π^T is *transposed push-down*.

Lift-up

Input $v_0, \ldots, v_{p-1} \dashv \mathbb{U}_{i-1}$
Output $v \dashv \mathbb{U}_i$ s.t. $v = v_0 + \cdots + v_{p-1} x_i^{p-1}$

1. Compute the linear form $\mathsf{Tr} \in \mathbb{U}_i^{D^*}$,
2. compute $\ell = (v_0 + \cdots + v_{p-1} x_i^{p-1}) \cdot \mathsf{Tr}$,
3. compute $P_v = \text{Push-down}^T(\ell)$,
4. compute $N_v(Z) = P_v(Z) \cdot \operatorname{rev}(Q_i)(Z) \mod Z^{p^i d - 1}$,
5. return $\operatorname{rev}(N_v)/Q_i' \mod Q_i$.

Speeding up some arithmetics

$\mathbb{U}_k$

p

$\mathbb{U}_{k-1}$

$\vdots$

$\mathbb{U}_2$

p

$\mathbb{U}_1$

p

$\mathbb{F}_q$

Galois action

$$\varphi : \begin{cases} G \times \mathbb{U}_i & \rightarrow \mathbb{U}_i \\ (\sigma, v) & \mapsto \sigma(v) \end{cases}$$

$$G := \mathsf{Gal}(\mathbb{U}_{i+1}/\mathbb{U}_i) \simeq \mathbb{Z}/p\mathbb{Z}$$

Speeding up some arithmetics

$\mathbb{U}_k$

$\mathbb{U}_{k-1}$

$\mathbb{U}_1$

$\mathbb{U}_0$

Divide and conquer

We improve some operations in $\mathbb{U}_i$

op(v)

Where it works

- traces,
- p-th roots,
- pseudotraces,
- inversion,
- Galois action,
- ...

Speeding up some arithmetics

$\mathbb{U}_k$

$\mathbb{U}_{k-1}$

$\mathbb{U}_1$

$\mathbb{U}_0$

Divide and conquer

We improve some operations in $\mathbb{U}_i$

- push-down the operands;

$$\mathrm{op}(v) \downarrow \quad v_0, \ldots, v_{p-1}$$

Where it works

- traces,
- p-th roots,
- pseudotraces,
- inversion,
- Galois action,
- ...

Speeding up some arithmetics

$\mathbb{U}_k$

$\mathbb{U}_{k-1}$

$\mathbb{U}_1$

$\mathbb{U}_0$

Divide and conquer

We improve some operations in $\mathbb{U}_i$

- push-down the operands;
- recursively solve p instances in $\mathbb{U}_{i-1}$;

op(v)

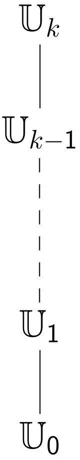
$\downarrow$

op(v_0), $\ldots$, op(v_{p-1})

Where it works

- traces,
- p-th roots,
- pseudotraces,
- inversion,
- Galois action,
- ...

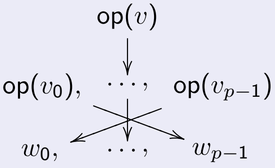
Speeding up some arithmetics



Divide and conquer

We improve some operations in $\mathbb{U}_i$

- push-down the operands;
- recursively solve p instances in $\mathbb{U}_{i-1}$;
- combine the results;



Where it works

- traces,
- p-th roots,
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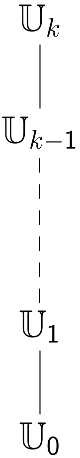
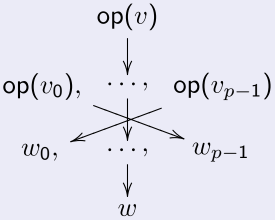
Speeding up some arithmetics



Divide and conquer

We improve some operations in $\mathbb{U}_i$

- push-down the operands;
- recursively solve p instances in $\mathbb{U}_{i-1}$;
- combine the results;
- lift-up.



Where it works

- traces,
- p-th roots,
- pseudotraces,
- inversion,
- Galois action,
- ...

Important application : Isomorphisms with generic towers

$\mathbb{U}_k$ — $\mathbb{U}_{k-1}$ — $\mathbb{U}_1$ — $\mathbb{U}_0$

Generic towers

- Let $(\alpha_0, \ldots, \alpha_{k-1})$ define a generic tower over $\mathbb{U}_0$,
- if we find an isomorphism we can bring fast arithmetics to it.

Computing the isomorphism [Couveignes '00]

Goal: factor $X^p - X - \alpha_i$ in U_{i+1}.

- Change of variables $X' = X - \mu$ s.t.
- $X'^p - X' - \alpha_i$ has a root in $\mathbb{U}_i$,
- Push-down, solve recursively, result is Δ,
- Lift-up Δ,
- return $\Delta + \mu$.

$\mathbb{U}'_k$ — $\mathbb{U}'_{k-1}$ — $\mathbb{U}'_1$ — $\mathbb{U}_0$

Outline

Implementation

Implementation in NTL + gf2x

Three types

- `GF2`: $p = 2$, FFT, bit optimisation,
- `zz_p`: $p < 2^{|\texttt{long}|}$, FFT, no bit-tricks,
- `ZZ_p`: generic p, like `zz_p` but slower.

Comparison to Magma

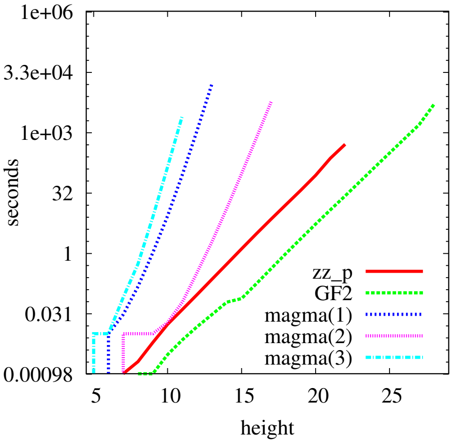
Three ways of handling field extensions

1. `quo<U|P>`: quotient of multivariate polynomial ring + Gröbner bases
2. `ext<k|P>`: field extension by $X^p - X - \alpha$, precomputed bases + multivariate
3. `ext<k|p>`: field extension of degree p, precomputed bases + multivariate

Benchmarks (on 14 AMD Opteron 2500)

Three modes

- $p = 2$, $d = 1$, height varying,
- p varying, $d = 1$, height $= 2$,
- $p = 5$, d varying, height $= 2$.


1e+06
3.3e+04
1e+03
32
1
0.031
0.00098
seconds
5
10
15
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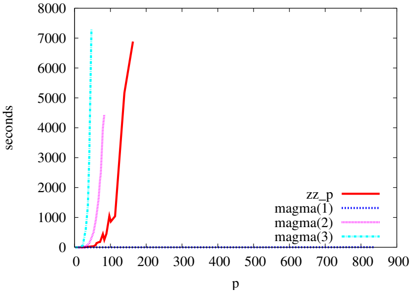
25
height
zz_p
GF2
magma(1)
magma(2)
magma(3)


8000
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6000
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2000
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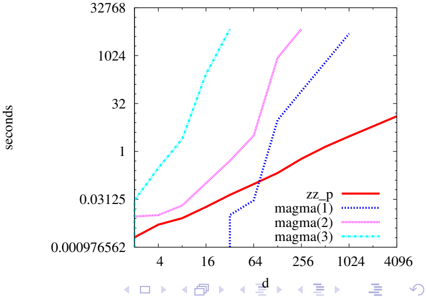
magma(3)


32768
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0.000976562
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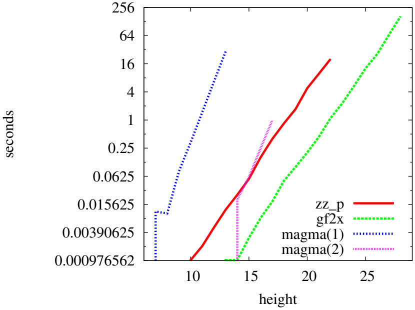
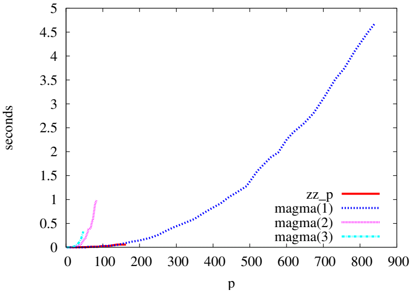
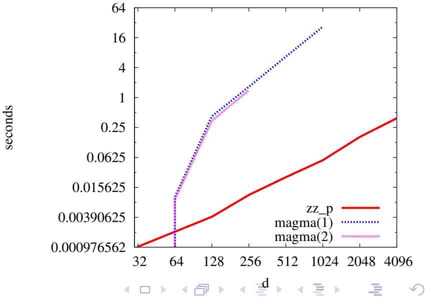
magma(2)
magma(3)

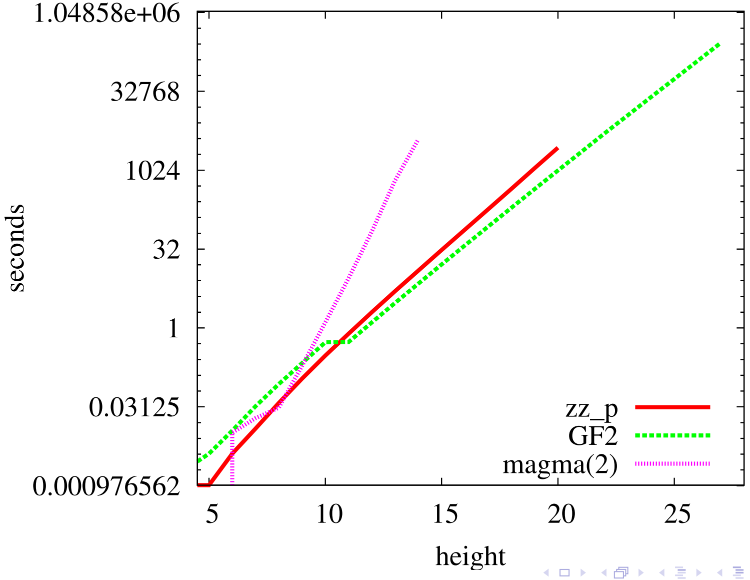
Multiplication





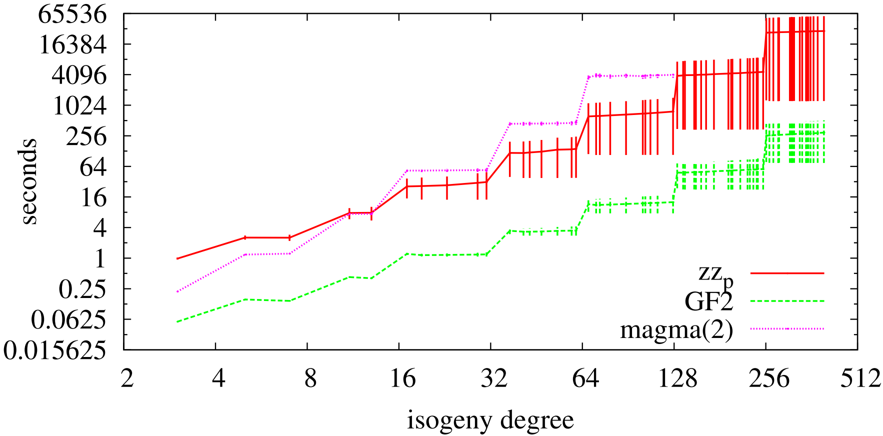


Isomorphism ([Couveignes '00] vs Magma)



Benchmarks on isogenies ([Couveignes '96])

Over $\mathbb{F}_{2^{101}}$, on an Intel Xeon E5430 Quad Core Processor 2.66GHz, 64GB ram



FAAST

These algorithms are packaged in a library

Download FAAST at
http://www.lix.polytechnique.fr/Labo/Luca.De-Feo/FAAST

We are currently writing an `spkg` for Sage.