Multiple Class G-Networks with iterated deletions

J-M. Fourneau, L. Kloul, F. Quessette

PRiSM
Université de Versailles
45, Av. des États-Unis
78000 Versailles, France

Abstract

We present a new type of multi-class generalized networks of queues with a steady-state product form solution. At its arrival into a queue, a negative customer (or a signal) starts an iteration. At each step of the iteration, a customer is deleted according to a probability which may depend on the type of customer. The iteration stops when the deletion fails. We study networks with multiple classes of positive customers, one class of signals and three service disciplines: FIFO, LIFO/PR and PS.

Keywords: Generalized networks, Product form solution, Multiple classes, Service disciplines, Iterated destructions.

1 Introduction

Since Gelenbe's seminal paper [12], Generalized networks have received considerable attention (see for instance [2, 15, 16, 18, 20] and references therein). Generalized networks (G-networks for short) consist of queues, ordinary customers and signals (which have been also denoted as negative customers). Signals interact with the customers at their arrival into a queue. But signals do not receive service, they are never queued and they disappear immediately. Here, we prefer to denote them as signals rather than as negative customers to emphasis that they are instantaneous.

The effect of a signal is limited to the interaction with the customers already queued at its arrival. But, at the completion of its service, a customer may join another queue as a signal. Thus, G-networks exhibit much more general synchronized transitions than Jackson's networks.

Several effects of signals have been studied and shown to preserve a product form solution. The first interaction, studied by Gelenbe, was the destruction of a customer [12]. Then, in [14] the product form was generalized to networks where a signal triggers a customer movement to a third queue. Gelenbe [13] and Henderson et al. [19] have independently studied batch destruction triggered by signals. If the batch size is infinite, a signal flushes out, with probability 1, the queue it enters. This effect, denoted as a catastrophe, has been studied by Chao in [4]. Several extensions have been proposed by Henderson, Taylor and Northcote to allows state-dependent batch movements and triggers [21]. However, most of these results only apply to single-class G-networks.

Multi-class G-networks with single deletion have been studied by Fourneau and Gelenbe [6],[8]. As usual with G-networks, only exponential service time distribution were allowed with class dependent service rate (except for FIFO queues). A generalization to Coxian services time distribution was proposed by Chao in [5] with a slightly different effect for a signal. Finally, a multi-class G-network with queue flushing and Processor-Sharing discipline was considered by Fourneau et al. [7].

In this paper, we generalize this last result. We consider G-networks with multiple classes of customers, one type of signals, three service disciplines: FIFO, LIFO/PR and PS, and more complex effects of signals. When a signal enters a queue, it starts to iterate on the deletion of customers in this queue. At each step, it tries to delete one customer. The customer is chosen according to the service discipline. The deletion succeeds following a Bernoulli process whose rate may depend on the class of the selected customer. If the deletion fails, the iteration stops.

Clearly, if the probability of success is 1, then an arriving signal flushes out a queue. As this probability may be class-dependent, we may obtain a much more complex behavior than queue flushing. Note however that, as the selection of the customer to be deleted must be consistent with the service discipline, deletion of arbitrary batches of customers do not occur with probability 1 even if the probability of success is 1. Due to the complexity of the deletion mechanism, we only consider exponential service time distribution. Indeed, the effect studied by Chao [5] to allow general service time becomes very complex when it is iterated.

G-networks were originally designed to model neural networks [11]. Signals and customers respectively represent inhibitory and excitatory signals in models of neural networks, while queue lengths represent the neurons input potentials. Recently, new applications of these networks have been proposed in the field of reliability and performability (see [9], [17] for instance). In these models, the deletion capabilities of signals is used to model breakdowns which cause the loss of some customers (or even all the customers) in a queue. As we add more complexity in the deletion mechanisms, we expect that more realistic models of systems with breakdown and failures will be achieved using multi-class G-networks with iterated deletions.

The paper is organized as follows. In the next section we present the model and prove the product form solution in section 3. For the sake of readability, the detailed proofs of some technical lemmas are postponed into an appendix. Examples are presented in section 4. Section 5 is devoted to stability, (i.e. the existence of solution for the flow equation). Like in usual G-networks, these equations are non linear and the same kind of technique is used to establish existence of a solution. We also study multi-class networks of queues with flushing which exhibit a more interesting behavior in term of stationarity constraints.

2 The Model

We consider a model of N queues, C classes of positive customers and one class of signals. The behavior of signals is described below. In each queue, the service discipline can be one of the following types:

- Type 1: first-in-first-out (FIFO),
- Type 2: processor sharing (PS),
- Type 4: last-in-last-out with preemptive resume priority (LIFO/PR).

The type number refers to the one used in BCMP theorem [1]. Type 3 described in [1] refers to service centers with an infinite number of servers. This type is not treated here since it does not fit in with the model.

Let us now describe the hypothesis on the queues. External arrivals to the queues follow independent Poisson processes for the customers and the signals. We note $\lambda_i^{(k)}$ the external arrival rate of class k customers to queue i and λ_i^- the external arrival rate of signals to this queue (remember that there is only one class of signals).

The service for a positive customer is as usual, but at the end of its service, it can be routed to another queue as a positive customer changing class with probability $P_{ij}^{+(k,l)}$, or as a signal with probability $P_{ij}^{-(k)}$. It can also be routed outside of the network with probability $d_i^{(k)}$. Note that it is of no interest to route a signal to the outside since this case is already considered in the routing of positive customers to the outside. We then have the following relation:

$$\forall i, k \quad \sum_{j=1}^{N} \sum_{l=1}^{C} P_{ij}^{+(k,l)} + \sum_{j=1}^{N} P_{ij}^{-(k)} + d_i^{(k)} = 1 \tag{1}$$

The behavior of the signal is the original part of this paper. If it arrives in an empty queue, it just disappears. If it arrives in a non empty queue i, its behavior depends on the service center type. In case of FIFO or LIFO service centers, it tries to destroy the customer in service with a probability $p_i^{(k)}$ of success and $1 - p_i^{(k)}$ of fail, where k is the class of the customer receiving service. If this destruction is successful, then it tries to destroy the next customer that should receive service in the queue, with a probability depending on its class. This mechanism is repeated until the destruction fails. In this case, the signal just disappears.

For PS service centers, the mechanism is in two steps. First, a customer class, say k is chosen according to the distribution of the customer number of each class in the queue. The probability for class k to be the one chosen is $\frac{x_i^k}{|\vec{x}_i|}$, where x_i^k is the number of class k customers and $|\vec{x}_i|$ the total number of customers in the queue. The second step is the trial of destruction with probability $p_i^{(k)}$ as in FIFO or LIFO. If the destruction fails, the signal disappears, if it succeeds the mechanism starts again.

Let us make some remarks about signals. First, the destruction can fail for the first trial and then the queue remains intact. Second, the deletion mechanism with signals is consistent with the service mechanism. Third, the transformation into signals, the routing to another queue and the deletion of positive customers are instantaneous events, and therefore, the Markov chain underlying this network has only to manage with the positive customers as it will be defined in the following section.

2.1 State Representation

We denote the state of the network by the vector $\vec{x} = (\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_N)$ where component $\vec{x}_i$ is the state of queue i. The state of each queue depends on the service type. $|\vec{x}_i|$ denotes the number of customers in the queue for all service disciplines.
For a FIFO or a LIFO service center, say i, the state is represented by a vector $\vec{x}_i$ whose length is the number of customers and whose component $r_i^{(k)}$ is the class of the kth customer in the queue. Furthermore, we denote by $r_{i,\infty}$ the class of the last customer in queue i. Note that the first customer in a FIFO or a LIFO queue is the one in service. In FIFO, when a customer arrives, it becomes the new last and in LIFO, it becomes the new first.

For a PS service center i, the state is represented by a vector of size K where each component $x_i^{(k)}$ is the number of class k customers in the queue.

2.2 Notations

Before we give the main theorem and its proof, we need to introduce some notations to carry out algebraic manipulations with some independence from the service center type.

First, we note $(\vec{x}_i + e_i^{(k)})$ the state of queue i with one more customer of class k. Note that this customer is the one in service for FIFO and LIFO service centers. In PS, it adds one to the kth component. The notation $(\vec{x}_i - e_i^{(k)})$ is used to describe the converse situation.

- The state-dependent service rates for customers of class k at service station i will be denoted by $M_{i,k}(\vec{x}_i)$ where $\vec{x}_i$ refers to the state of the service station i. From the definition of the service rates $\mu_i^{(k)}$ and according to the service center type, we have:
  - FIFO/LIFO: $M_{i,k}(\vec{x}_i) = \mu_i^{(k)} \mathbb{1}_{\{r_{i,1}=k\}}$
  - PS: $M_{i,k}(\vec{x}_i) = \mu_i^{(k)} \frac{x_i^k}{|\vec{x}_i|} \mathbb{1}_{\{|\vec{x}_i|>0\}}$
- We denote by $A_{i,k}(x_i)$ the condition which shows that it is possible to be in state $\vec{x}_i$ after the arrival of a positive customer of class k:
  - FIFO: $A_{i,k}(\vec{x}_i) = \mathbb{1}_{\{r_{i,\infty}=k\}}$
  - LIFO: $A_{i,k}(\vec{x}_i) = \mathbb{1}_{\{r_{i,1}=k\}}$
  - PS: $A_{i,k}(\vec{x}_i) = \mathbb{1}_{\{x_i^{(k)}>0\}}$
- $N_{i,k}$ is the probability that, given the system in state $\vec{x}_i$ and the customer selected by an incoming signal of type k, the first deletion is a success. As a consequence, $N_{i,k}$ denotes the probability that we leave state $\vec{x}_i$ due to an arrival of a signal which selects a class k customer and succeeds its deletion.
  - FIFO: $N_{i,k}(\vec{x}_i) = p_i^{(k)} \mathbb{1}_{\{r_{i,1}=k\}}$ where $\forall\, k \quad p_i^{(k)} = p_i$
  - LIFO: $N_{i,k}(\vec{x}_i) = p_i^{(k)} \mathbb{1}_{\{r_{i,1}=k\}}$
  - PS: $N_{i,k}(\vec{x}_i) = p_i^{(k)} \frac{x_i^k}{|\vec{x}_i|} \mathbb{1}_{\{|\vec{x}_i|>0\}}$
- $Z_i(\vec{x}_i, \vec{y}_i)$ denotes the probability to be in state $\vec{x}_i$ after the arrival of a signal which destroys $|\vec{y}_i|$ positive customers. $|\vec{y}_i|$ is a vector which component y_i^k denotes the total number of positive customers of class k deleted by the signal.
  - FIFO:
    $Z_i(\vec{x}_i, \vec{0}) = \mathbb{1}_{\{|\vec{x}_i|=0\}} + \sum_{k=1}^{C} (1 - p_i^{(k)}) \mathbb{1}_{\{r_{i,1}=k\}}$
    $Z_i(\vec{x}_i, \vec{y}_i + e_i^{(k)}) = Z_i(\vec{x}_i, \vec{y}_i) p_i^{(k)}$
  - LIFO:
    $Z_i(\vec{x}_i, \vec{0}) = \mathbb{1}_{\{|\vec{x}_i|=0\}} + \sum_{k=1}^{C} (1 - p_i^{(k)}) \mathbb{1}_{\{r_{i,1}=k\}}$
    $Z_i(\vec{x}_i, \vec{y}_i + e_i^{(k)}) = Z_i(\vec{x}_i, \vec{y}_i) p_i^{(k)}$
  - PS:
    $Z_i(\vec{x}_i, \vec{0}) = \mathbb{1}_{\{|\vec{x}_i|=0\}} + \sum_{k=1}^{C} (1 - p_i^{(k)}) \frac{x_i^k}{|\vec{x}_i|} \mathbb{1}_{\{|\vec{x}_i|>0\}}$
    $Z_i(\vec{x}_i, \vec{y}_i + e_i^{(k)}) = Z_i(\vec{x}_i, \vec{y}_i) \frac{x_i^{(k)} + y_i^k + 1}{|\vec{x}_i + \vec{y}_i| + 1} p_i^{(k)}$

3 Product Form Theorem

Let $\Pi(\vec{x})$ be the stationary probability distribution of the network state if it exists. The following establishes the existence of a product form solution of the network type being considered.

Theorem 1 *: Consider an arbitrary open G-network with C classes of positive customers and a single class of signals. If the system of non-linear equations:*

$$\rho_i^{(k)} = \frac{\lambda_i^{(k)} + \sum_{j=1}^{N} \sum_{l=1}^{C} \mu_j^{(l)} \rho_j^{(l)} P_{ji}^{+(l,k)}}{\mu_i^{(k)} + \left[\lambda_i^{-} + \sum_{j=1}^{N} \sum_{l=1}^{C} \mu_j^{(l)} \rho_j^{(l)} P_{ji}^{-(l)}\right] S_i p_i^{(k)}} \tag{2}$$

with $S_i = \sum_{n=0}^{+\infty} \left(\sum_{k=1}^{C} \rho_i^{(k)} p_i^{(k)} \right)^n$ *has a positive solution such that for each station i:*

$$\sum_{k=1}^{C} \rho_i^{(k)} p_i^{(k)} < 1 \tag{3}$$

then the stationary distribution of the network state exists and has a product form:

$$\Pi(\vec{x}) = G \prod_{i=1}^{N} g_i(\vec{x}_i) \tag{4}$$

where $g_i(x_i)$ *depends on the type of the service station i as follows:*

- *FIFO/LIFO:*

$$g_i(\vec{x}_i) = \prod_{n=1}^{|\vec{x}_i|} \rho_i^{r_{i,n}}$$

- *PS:*

$$g_i(\vec{x}_i) = |\vec{x}_i|! \prod_{k=1}^{C} \frac{(\rho_i^k)^{x_{i,k}}}{x_i^k!}$$

and G is the normalization constant.

Proof of theorem 1: The proof consists mainly of algebraic manipulations of the following Chapmann-Kolmogorov equation:

$$
\begin{aligned}
\Pi(\vec{x}) \sum_{i=1}^{N} \sum_{k=1}^{C} \left[\lambda_i^{(k)} + M_{i,k}(\vec{x}_i) + \lambda_i^- N_{i,k}(\vec{x}_i)\right] = & \\
& \sum_{i=1}^{N} \sum_{k=1}^{C} \lambda_i^{(k)} A_{i,k}(\vec{x}_i) \Pi(\vec{x} - e_i^{(k)}) \\
+ & \sum_{i=1}^{N} \sum_{k=1}^{C} M_{i,k}(\vec{x} + e_i^{(k)}) d_i^{(k)} \Pi(\vec{x} + e_i^{(k)}) \\
+ & \sum_{i=1}^{N} \sum_{\vec{y_i}/|\vec{y_i}|>0} \lambda_i^- Z_i(\vec{x}_i, \vec{y_i}) \Pi(\vec{x} + \vec{y_i}) \\
+ & \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{C} \sum_{l=1}^{C} M_{i,k}(\vec{x} + e_i^{(k)} - e_j^{(l)}) A_{j,l}(\vec{x}_j) P_{ij}^{+(k,l)} \Pi(\vec{x} + e_i^{(k)} - e_j^{(l)}) \\
+ & \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{\vec{y_j}/|\vec{y_j}|>=0} M_{i,k}(\vec{x} + e_i^{(k)} + \vec{y_j}) Z_j(\vec{x}_j, \vec{y_j}) P_{ij}^{-(k)} \Pi(\vec{x} + e_i^{(k)} + \vec{y_j})
\end{aligned}
$$

For the sake of readability, the proof of the theorem is in appendix A. We use a proof based on algebraic manipulations rather than a probabilistic method as the one proposed by Chao and Pinedo in [5]. They consider very simple G-networks (single class networks with positive and negative customers) and they identify negative customers as services. When a customer begins service, it triggers a race between 2 servers (one with rate μ and the other one with rate Λ_i^-). They prove that these simple G-networks are quasi-reversible and hence have a product form solution. However, their method, quite "elegant" in single class and single deletion frameworks becomes quite difficult with multiple classes and multiple deletions networks.

Now, we can compute by some trivial algebraic manipulations the steady-state distribution of the number of customers of each class in each queue. Let $\vec{u_i}$ be the vector whose components are u_i^k, the number of customers of class k in station i. Let $\vec{u}$ be the vector of vectors $\vec{u_i}$.

Theorem 2 *If system 2 has a solution, then the steady-state distribution $\pi(\vec{x})$ is given by:*

$$\pi(\vec{u}) = \prod_{i=1}^{N} h_i(\vec{u_i})$$

where the marginal probabilities $h_i(\vec{x}_i)$ have the following form:

$$h_i(\vec{u_i}) = \left(1 - \sum_{k=1}^{C}\right) |\vec{u_i}|! \prod_{k=1}^{C} \frac{(\rho_{i,k})^{u_{i,k}}}{u_{i,k}!}$$

We omit the proof of this theorem since the result is quite usual for multiple classes queues.

4 Examples and Comments

We present now some examples to explain what kind of transitions takes place in our networks. In particular, it must be clear that a signal cannot delete all customers of a given class with probability one for an arbitrary state x.

4.1 Example 1: 2 classes of customers

Consider a queueing network with 2 classes of positive customers ($C = 2$). Figure 1 illustrates a queue, of such a network. In this queue, 8 customers wait for service. These customers are designed by their class number c_j, $j = 1, 2$. When a signal arrives in the queue, the deletion probability of a queued customer depends on the customer class as follows: $p^{c_1} = 1$ and $p^{c_2} = 0$. As it was previously explained, the deletion of a positive customer depends also on the service type. Let's see now how our destruction mechanism works according to the defined probabilities and the service center types.

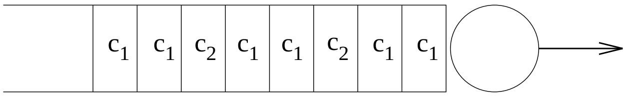

Figure 1: A queue with 2 classes of customers.

LIFO: Since the deletion probability of a customer of class c_1 is $p^{c_1} = 1$, when a signal arrives to the queue, it destroys the first customer in it, which is a customer of class c_1. After that, it deletes the second customer since it is also of class c_1. Then it stops, because the third customer in the queue is of class c_2 and that the deletion probability of customers of this class is $p^{c_2} = 0$.

PS: For a PS queue, the corresponding state is ($6c_1$, $2c_2$). When a signal arrives to the queue, the positive customer to destroy is first chosen with probability $\frac{x_i^{c_j}}{\bar{x}_i}$ (for our example, this probability is respectively $\frac{3}{4}$ and $\frac{1}{4}$). If the customer chosen is of class c_1, it will be destroyed with probability $p^{c_1} = 1$. So, the destruction mechanism may continue. If the customer chosen is of class c_2, because of the deletion probability of such customers, the signal disappears and the deletion mechanism stops. Thus, the arrival of a signal will lead to the states and according to the probabilities given by the following table:

States	*Probabilities*
$6c_1, 2c_2$	$2/8$
$5c_1, 2c_2$	$6/8 \times 2/7$
$4c_1, 2c_2$	$6/8 \times 5/7 \times 2/6$
$3c_1, 2c_2$	$6/8 \times 5/7 \times 4/6 \times 2/5$
$2c_1, 2c_2$	$6/8 \times 5/7 \times 4/6 \times 3/5 \times 2/4$
$1c_1, 2c_2$	$6/8 \times 5/7 \times 4/6 \times 3/5 \times 2/4 \times 2/3$
$2c_2$	$6/8 \times 5/7 \times 4/6 \times 3/5 \times 2/4 \times 1/3$

The probabilities are written as a product of fractions rather than computed to emphasis the iteration process for the customer deletion.

4.2 Example 2: 3 classes of customers

Consider now the queue in Figure 2. In this queue, we can see this time 3 classes of customers: c_1, c_2 and c_3. When a signal arrives in the queue, the deletion probability of a queued customer depends on the customer class as follows: $p^{c_1} = 1$, $p^{c_2} = 0.2$ and $p^{c_3} = 0.4$. Let's see now how our destruction mechanism works according to the defined probabilities and the service center types.

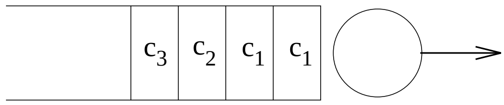

Figure 2: A queue with 3 classes of customers.

LIFO: Since the deletion probability of a customer of class c_1 is $p^{c_1} = 1$, when a signal arrives to the queue, it destroys the first customer in it, which is a customer of class c_1. After that, it deletes the second customer since it is also of class c_1. The third customer in the queue is of class c_2. If we look at the deletion probability of customers of this class ($p^{c_2} = 0.2$), we can easily understand that the deletion mechanism may fail. If the signal fails in deleting the class c_2 customer, the deletion mechanism stops immediately and the signal disappears. In the case of a success, the deletion mechanism will continue.

Just like for class c_2 customers, and because of the destruction probability is not one ($p^{c_3} = 0.4$), the deletion of class c_3 customers may fail. Thus, the arrival of a signal into that queue may trigger a transition into only 3 states according to the following probability distribution:

States	*Probabilities*
c_3, c_2	0.8
c_3	0.2×0.6
$\emptyset$	0.2×0.4

PS: The initial state is denoted as ($2c_1$, $1c_2$, $1c_3$). Several transitions may be triggered due to the iterated sampling of the customer and the probabilities of success for each class of customers. These transitions and their probability distribution are gathered into the following table:

States	*Probabilities*
$2c_1, 1c_2, 1c_3$	105/300
$1c_1, 1c_2, 1c_3$	70/300
$2c_1, 1c_2$	8/300
$1c_1, 1c_2$	16/300
$2c_1, 1c_3$	3/300
$1c_1, 1c_3$	6/300
$1c_2, 1c_3$	35/300
$1c_2$	24/300
$1c_3$	9/300
$\emptyset$	24/300

5 Stability of the fixed point system

As system (2) is non linear, the existence of a solution to this fixed point system is not obvious, except for the feed-forward networks. In the following, we prove that under simple assumptions, there exists a solution to the traffic equation (2). First, let us define the open topology for G-network.

Definition 1 *A network with N queues is called an* ***open G-network*** *if the matrix P^+ does not contain any ergodic classes.*

This assumption was clearly stated by Gelenbe and Schassberger in [16] to prove the existence of a solution to the flow equations.

In this section, we assume that $\lambda_i^- > 0$, $\forall i$. This rather restricted assumption is necessary for technical details (the continuity on the domain since the sum S_i may be convergent or divergent). The use of a more general assumption on the structure of the network remains an open problem.

5.1 Stability Conditions

Let F be an operator on $(\mathbb{R}^+)^{N\times C}$ defined by its components $F_i^{(k)}$ such that:

$$\begin{cases} F_i^{(k)}(\rho) = \dfrac{\lambda_i^{(k)} + \displaystyle\sum_{j=1}^{N}\sum_{l=1}^{C} \rho_j^{(l)} \mu_j^{(l)} P_{ji}^{+(l,k)}}{\mu_i^{(k)} + \left[\lambda_i^- + \displaystyle\sum_{j=1}^{N}\sum_{l=1}^{C} \rho_j^{(l)} \mu_j^{(l)} P_{j,i}^{-(l)}\right] \dfrac{p_i^{(k)}}{1-\alpha_i}} & when \quad \alpha_i < 1 \\ F_i^{(k)}(\rho) = 0 & when \quad \alpha_i \geq 1 \end{cases}$$

where $\alpha_i = \sum_{l=1}^{C} \rho_i^{(l)} p_i^{(l)}$.

If $\lambda_i^- > 0$ then clearly $\lambda_i^- + \sum_{j=1}^{N}\sum_{l=1}^{C} \mu_j^{(l)} \rho_j^{(l)} P_{j,i}^{-(l)} > 0$ and F is an extension by continuity of system (2). Thus, the solutions of system (2) are also solutions of the fixed point equation $\rho = F(\rho)$. Furthermore, the solutions of $\rho = F(\rho)$ which are not solutions of system (2) do not satisfy the constraint $\sum_{l=1}^{C} \rho_i^{(l)} p_i^{(l)} < 1$ and they are useless to characterize the steady-state distribution of the network.

Now, we investigate the existence of the fixed point of F, which leads to the existence of the fixed point of system (2).

Theorem 3 *For an open G-network with $\lambda_i^- > 0$ for all i, system (2) has a solution in $(\mathbb{R}^+)^{N\times C}$.*

Proof: Since a fixed point of F is also a solution to system (2), using Brouwer's theorem [10], we prove here that F has a fixed point. We define a new operator, say H, on $(\mathbb{R}^+)^{N\times C}$ by its components:

$$H_i^{(k)}(\rho) = \lambda_i^{(k)} + \sum_{j=1}^{N}\sum_{l=1}^{C} \rho_j^{(l)} \mu_j^{(l)} P_{ji}^{+(l,k)}.$$

Using a matrix formulation, the definition of H may be restated as $H(\rho) = \lambda + \rho P^+$ where λ is a vector which components are the arrival rates $\lambda_i^{(k)}$. Since we are in an open G-network, the routing matrix P^+ is transient, and therefore $(I - P^+)$ is invertible and H has a fixed point $\hat{\rho}$ defined by:

$$\hat{\rho} = \lambda(I - P^+).$$

Now, we define S as a subset of $(I\!R^+)^{N\times C}$ as follows:

$$S = \{\rho \in (I\!R^+)^{N\times C} : 0 \preceq \rho \preceq \hat{\rho}\}.$$

Clearly, S is compact and convex and the interior of S is not empty, since $\lambda \succ 0$ for an open G-network.

Clearly, for all ρ in S, we have $F(\rho) \preceq H(\rho)$ and for all ρ_1, ρ_2 in S, such that $\rho_1 \preceq \rho_2$ component by component, we have $H(\rho_1) \preceq H(\rho_2)$, so for all ρ in S we have:

$$F(\rho) \preceq H(\rho) \preceq H(\hat{\rho}) = \hat{\rho}$$

and then $F(S) \subseteq S$.

Since S is compact convex and has a non empty interior, since F is continuous and $F(S) \subseteq S$, then F satisfies assumptions of Brouwer's theorem [10]. Thus, F has a fixed point.

□

5.2 Stationarity of the system when $p_i^{(k)} = 1$

We study now networks where $p_i^{(k)} = 1$. We prove that if a solution exists to fixed point system (2) then, under general structural conditions on the network, the solution satisfies the stationarity constraint. Therefore the Markov chain is positive recurrent and has a product form solution. This structural condition is the same as the one used to state the system stability ($\lambda_i^- > 0$). This is equivalent to say that signals may enter queue i from outside the network. It should be noted that the results bellow are obtained for any particular value of the arrival rates and the service rates.

In the following, we assume that a solution exists to fixed point system (2). Clearly, the chain has a stationarity product form solution if and only if S_i is finite. So, we have to prove that all the solutions to fixed point system (2) are such that S_i is finite.

Theorem 4 *If for all i, we have $\lambda_i^- > 0$ then the network has a stationary product form solution, for any particular positive values of the arrival rates and service rates.*

Proof: Stating $\lambda_i^- > 0$ implies that $\lambda_i^- + \sum_{j=1}^{N}\sum_{l=1}^{C} \mu_j^{(l)} \rho_j^{(l)} P_{j,i}^{-(l)} > 0$.

According to this statement, if we assume that $S_i = \sum_{l=1}^{C} p_i^{(l)} \rho_i^{(l)} \geq 1$, we will then have $\rho_i^{(k)} = 0$, for all k. However, we know that in this case, the function $F_i^{(k)} = 0$ and as this function, system(2) does not have a fixed point. Then we must have $S_i = \sum_{l=1}^{C} p_i^{(l)} \rho_i^{(l)} < 1$. Thus, according to theorem 1, the network always has a steady-state product form distribution.

Remark: The assumption necessary to prove that the solution to the fixed point system satisfies the stationarity constraint may be more general than the one used here. Indeed, a simpler structural condition can be used. This condition is equivalent to say that signal may enter a queue i either from outside the network or from another queue j. This assumption has been used in [7].

5.3 Asymptotic Analysis

For each queue i of the network, we study two performance measures, the mean number of customers N_i and the average sojourn time W_i when the arrival rate grows to infinity. These measures concern only the positive customers in the network since the signals disappear instantaneously.

We consider networks such that $p_i^{(k)} = 1$, for all $k = 1, .., C$, and:

- $\forall i \in [1..N]$, $\exists k \in [1..C]$ such that $\lambda_i^{(k)} > 0$,
- $\forall i \in [1..N]$, $\lambda_i^- > 0$,
- there exists a topological order on the queues such that the matrix P^- can be upper triangular, and we assume that the queues are numbered according to this order.

The mean number of customers in a queue is obtained from the marginal distribution as usual:

$$N_i = \frac{\Omega_i}{1 - \Omega_i}$$

where $\Omega_i = \sum_{k=1}^{C} \rho_i^{(k)}$ is the total load at station i. Since the mean number of customers in the system depends only on the positive customers arrivals and departures (service completion or destruction by a signal), we can apply Little's formula to obtain the average sojourn time:

$$W_i = \frac{N_i}{\sum_{k=1}^{C} \lambda_i^{(k)}}$$

In the first part of the analysis, we study what we will call the *input queues* which are the queues of the network that receive signals from the outside world only. Then, using the results obtained for the input queues, we generalize the study to include the queues which receive signals from both, the outside world and the queues which precede them, according to the topological ordering.

In the case of an input queue i, system (2) becomes:

$$\rho_i^{(k)} = \frac{\lambda_i^{(k)} + \sum_{j=1}^{N} \sum_{l=1}^{C} \rho_j^{(l)} \mu_j^{(l)} P_{ji}^{+(l,k)}}{\mu_i^{(k)} + \lambda_i^- \frac{1}{1 - \sum_{l=1}^{C} \rho_i^{(l)}}}$$

Assume now that for all classes, positive customers arrive with a rate $\lambda_i^{(k)} = \Lambda_i r_i^{(k)}$ where $r_i^{(k)}$ is the constant ratio of class k customers that arrive to queue i with $\sum_{k=1}^{C} r_i^{(k)} = 1$. After some algebraic manipulations and since $\sum_{j=1}^{N} \sum_{l=1}^{C} \rho_j^{(l)} \mu_j^{(l)} P_{ji}^{+(l,k)}$ is bounded, taking the truncated third order series approximation of Ω_i, we obtain:

$$\lim_{\Lambda_i \to +\infty} \Omega_i = \lim_{\Lambda_i \to +\infty} \left[1 - \frac{\lambda_i^-}{\Lambda_i} + \frac{(\lambda_i^-)^2}{\Lambda_i^2} - \frac{\lambda_i^-}{\Lambda_i^2} \sum_{k=1}^{C} \left(\sum_{j=1}^{N} \sum_{l=1}^{C} \rho_j^{(l)} \mu_j^{(l)} P_{ji}^{+(l,k)} + r_i^{(k)} \mu_i^{(k)} \right) \right] = 1$$

Clearly when Λ_i approaches infinity, the load approaches 1 from below. This value of load is never reached because a part of the positive customers is never served, it is destroyed by the

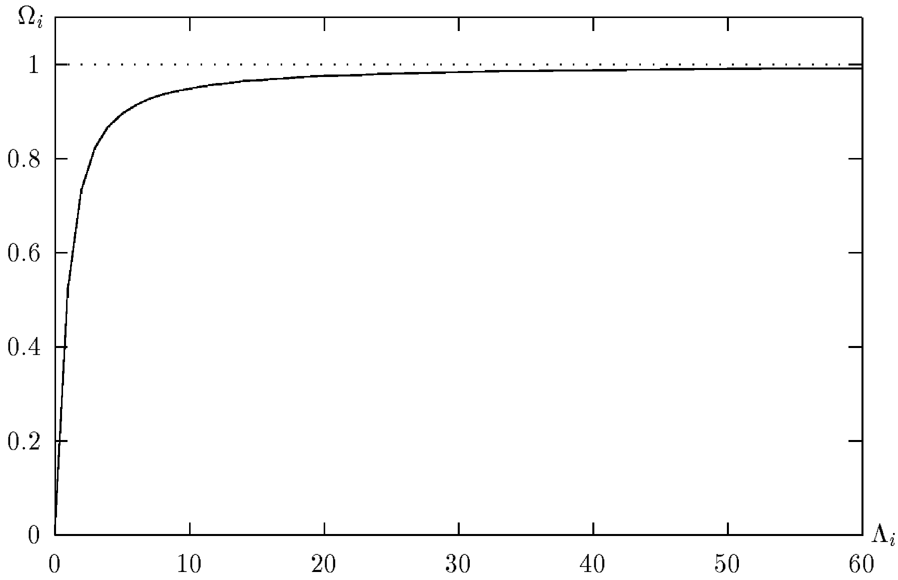

Figure 3: Behavior of the load Ω_i for $p = 2$ and $\lambda_i^- = 0.5$.

signals. Figure 3 shows the behavior of the load Ω_i in the case where the positive customers arrive from outside only ($\sum_{j=1}^{N} \sum_{l=1}^{C} \rho_j^{(l)} \mu_j^{(l)} P_{ji}^{+(l,k)} = 0$).

When Λ_i approaches infinity, the behavior of the mean number of customers N_i and the average sojourn time W_i are respectively given by:

$$N_i = \frac{1}{\lambda_i^-} \left[\sum_{k=1}^{C} \left(\sum_{j=1}^{N} \sum_{l=1}^{C} \rho_j^{(l)} \mu_j^{(l)} P_{ji}^{+(l,k)} - r_i^{(k)} \mu_i^{(k)} \right) \right] + \Lambda_i \frac{1}{\lambda_i^-} + o(1/\Lambda_i)$$

and

$$\lim_{\Lambda_i \to +\infty} W_i = \frac{1}{\lambda_i^-}$$

We can note the asymptotic behavior (Figure 4) of the average sojourn time W_i which, when Λ_i approaches infinity, approaches a value which is inversely proportional to the external arrival rate of signals to queue i. This can be easily explained, because when Λ_i tends to infinity, it means that there are more arrivals of positive customers than service completions. At that moment, W_i depends mostly only on the arrival rate of signals. The more important this rate is, the smaller the waiting time is. Negative customers can also be considered as a second service process. Figure 5 shows also the behavior of the mean number of customers in queue i.

□

Now consider system (2) for the queues that receive signals from both the outside and the preceding queues:

$$\rho_i^{(k)} = \frac{\lambda_i^{(k)} + \sum_{j=1}^{N} \sum_{l=1}^{C} \rho_j^{(l)} \mu_j^{(l)} P_{ji}^{+(l,k)}}{\mu_i^{(k)} + (\lambda_i^- + \sum_{j=1}^{i-1} \sum_{l=1}^{C} \rho_j^{(l)} \mu_j^{(l)} P_{j,i}^{-(l)}) \frac{1}{1 - \sum_{l=1}^{C} \rho_i^{(l)}}}$$

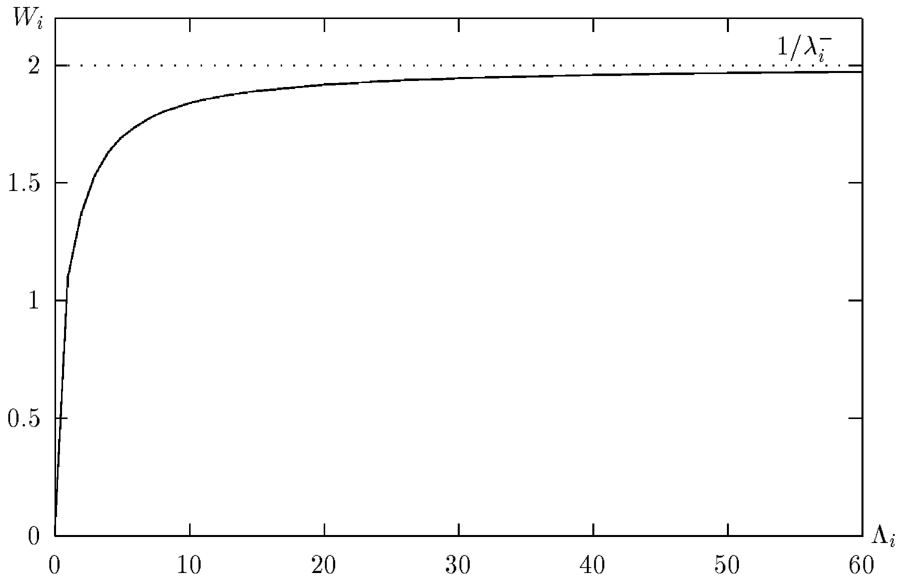

Figure 4: Behavior of the mean sojourn time W_i for $C = 2$ and $\lambda_i^- = 0.5$.

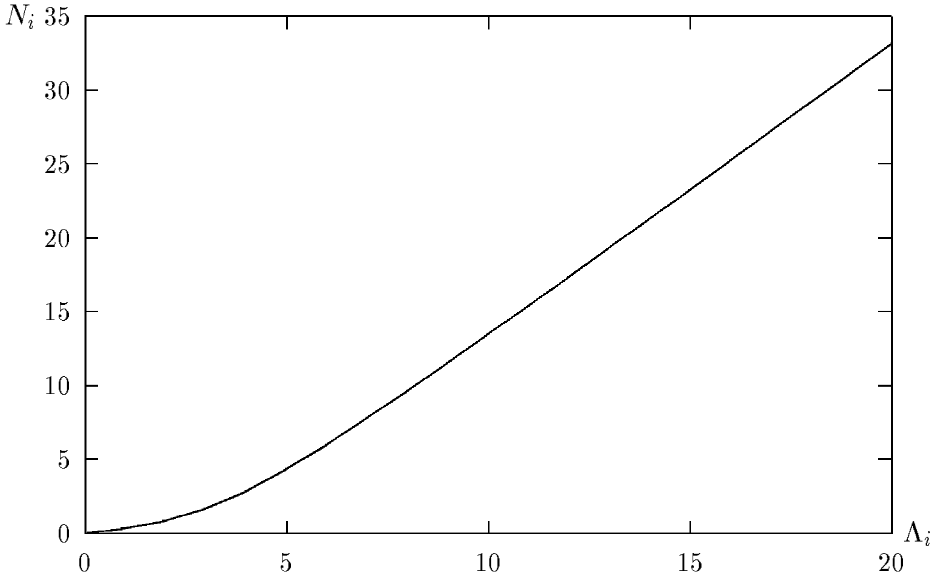

Figure 5: Behavior of the customers mean number N_i for $C = 2$ and $\lambda_i^- = 0.5$.

After some algebraic manipulations and taking the third order series approximation of the total load $\Omega_i = \sum_{k=1}^{C} \rho_i^{(k)}$, we obtain:

$$\begin{aligned}
\lim_{\Lambda_i \to +\infty} \Omega_i &= \lim_{\Lambda_i \to +\infty} \left[1 - \frac{\lambda_i^- + \sum_{j=1}^{i-1} \sum_{l=1}^{C} \rho_j^{(l)} \mu_j^{(l)} P_{j,i}^{-(l)}}{\Lambda_i} + \frac{(\lambda_i^- + \sum_{j=1}^{i-1} \sum_{l=1}^{C} \rho_j^{(l)} \mu_j^{(l)} P_{j,i}^{-(l)})^2}{\Lambda_i^2} \right. \\
&\quad \left. \times \left(1 - \frac{\sum_{k=1}^{C} \left(r_i^{(k)} \mu_i^{(k)} + \sum_{j=1}^{N} \sum_{l=1}^{C} \rho_j^{(l)} \mu_j^{(l)} P_{ji}^{+(l,k)} \right)}{\lambda_i^- + \sum_{j=1}^{i-1} \sum_{l=1}^{C} \rho_j^{(l)} \mu_j^{(l)} P_{j,i}^{-(l)}} \right) \right] \\
&= 1
\end{aligned}$$

If we look at the average number of customers in a queue i when this same rate approaches infinity, we have:

$$N_i = \Lambda_i \frac{1}{\lambda_i^- + \sum_{j=1}^{i-1} \sum_{l=1}^{C} \rho_j^{(l)} \mu_j^{(l)} P_{j,i}^{-(l)}} + o(1)$$

When the arrival rate Λ_i approaches infinity, the mean sojourn time W_i never exceeds a load-independent asymptotic value:

$$\lim_{\Lambda_i \to +\infty} W_i = \frac{1}{\lambda_i^- + \sum_{j=1}^{i-1} \sum_{l=1}^{C} \rho_j^{(l)} \mu_j^{(l)} P_{ji}^{-(l)}}$$

As for the input queues, when Λ_i tends to infinity, the mean sojourn time tends to a value which is inversely proportional to the signal arrival rate, but in this case, the arrival rate is composed of the external arrival rate and the arrival rate from the queues which precede queue i.

□

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A Technical Appendix

Before proceeding with the theorem proof, we need to introduce some technical lemmas.

Lemma 1 *: For any type of service center, the following relation holds:*

$$M_{i,k}(\vec{x}+e_i^{(k)})\frac{\Pi(\vec{x}+e_i^{(k)})}{\Pi(\vec{x})} = \mu_i^{(k)}\rho_i^{(k)}$$

Proof: The proof is purely algebraic.

Lemma 2 *: If we set $Z_i(\vec{x}_i,\vec{0}) = 1 - W_i(\vec{x}_i)$, $W_i(\vec{x}_i)$ will depend on the service station type as follows:*

- *FIFO:* $W_i(\vec{x}_i) = \sum_{k=1}^{C} p_i^{(k)} \mathbb{1}_{\{r_{i,1}=k\}}$
- *LIFO:* $W_i(\vec{x}_i) = \sum_{k=1}^{C} p_i^{(k)} \mathbb{1}_{\{r_{i,1}=k\}}$
- *PS:* $W_i(\vec{x}_i) = \sum_{k=1}^{C} p_i^{(k)} \frac{x_i^k}{|\vec{x}_i|} \mathbb{1}_{\{|\vec{x}_i|>0\}}$

Proof: The proof consists of algebraic manipulations.

Lemma 3 *: For any type of service center, the following relation holds:*

$$\sum_{\vec{y_i}/|\vec{y_i}|>0} Z_i(\vec{x}_i,\vec{y_i})\frac{\Pi(\vec{x}+\vec{y_i})}{\Pi(\vec{x})} = (S_i-1)Z_i(\vec{x}_i,\vec{0})$$

where $S_i = \sum_{n=0}^{+\infty}\left(\sum_{k=1}^{C}\rho_i^{(k)}p_i^{(k)}\right)^n$.

Proof:

$$\begin{aligned}\sum_{\vec{y_i}/|\vec{y_i}|>0} Z_i(\vec{x}_i,\vec{y_i})\frac{\Pi(\vec{x}+\vec{y_i})}{\Pi(\vec{x})} &= \textstyle\sum_{\vec{y_i}/|\vec{y_i}|>0} Z_i(\vec{x}_i,\vec{y_i})\frac{g_i(\vec{x}_i+\vec{y_i})}{g_i(\vec{x}_i)}\\ &= \sum_{R=1}^{+\infty}\sum_{|\vec{v}|=R} Z_i(\vec{x}_i,\vec{v})\frac{g_i(\vec{x}_i+\vec{v})}{g_i(\vec{x}_i)}\end{aligned}$$

Using note A, for all type of service center, we have:

$$\begin{aligned}\sum_{\vec{y_i}/|\vec{y_i}|>0} Z_i(\vec{x}_i,\vec{y_i})\frac{\Pi(\vec{x}+\vec{y_i})}{\Pi(\vec{x})} &= \sum_{R=1}^{+\infty}\left[\left(\sum_{k=1}^{C}p_i^{(k)}\rho_i^{(k)}\right)\sum_{|\vec{v}|=R-1} Z_i(\vec{x}_i,\vec{v})\frac{g_i(\vec{x}_i+\vec{v})}{g_i(\vec{x}_i)}\right]\\ &= \sum_{R=1}^{+\infty}\left[\left(\sum_{k=1}^{C}p_i^{(k)}\rho_i^{(k)}\right)^R\sum_{|\vec{v}|=0} Z_i(\vec{x}_i,\vec{0})\frac{g_i(\vec{x}_i)}{g_i(\vec{x}_i)}\right]\\ &= Z_i(\vec{x}_i,\vec{0})\sum_{R=1}^{+\infty}\left(\sum_{k=1}^{C}p_i^{(k)}\rho_i^{(k)}\right)^R\\ &= Z_i(\vec{x}_i,\vec{0})\left[\sum_{R=0}^{+\infty}\left(\sum_{k=1}^{C}p_i^{(k)}\rho_i^{(k)}\right)^R - 1\right]\\ &= Z_i(\vec{x}_i,\vec{0})(S_i-1)\end{aligned}$$

Remark: For any type of service center, the following relation is satisfied:

$$Z_i(\vec{x}_i, \vec{y_i} + e_i^{(k)}) \frac{g_i(x_i + y_i + e_i^{(k)})}{g_i(x_i)} = Z_i(\vec{x}_i, \vec{y_i}) \frac{g_i(x_i + y_i)}{g_i(x_i)} \rho_i^{(k)} p_i^{(k)}$$

□

Lemma 4 *: For any type of service center, the following relation holds:*

$$\sum_{\vec{y_j}/|\vec{y_j}|>=0} M_{i,k}(\vec{x} + e_i^{(k)} + \vec{y_j}) Z_j(\vec{x}_j, \vec{y_j}) \frac{\Pi(\vec{x} + e_i^{(k)} + \vec{y_j})}{\Pi(\vec{x})} = \mu_i^{(k)} \rho_i^{(k)} Sj Z_j(\vec{x}_j, \vec{0})$$

Proof: We set:

$$\Delta_{i,k} = \sum_{\vec{y_j}/|\vec{y_j}|>=0} M_{i,k}(\vec{x} + e_i^{(k)} + \vec{y_j}) Z_j(\vec{x}, \vec{y_j}) \frac{\Pi(\vec{x} + e_i^{(k)} + \vec{y_j})}{\Pi(\vec{x})}$$

This expression can be rewritten as follows:

$$\begin{aligned} \Delta_{i,k} &= \sum_{\vec{y_j}/|\vec{y_j}|>=0} M_{i,k}(\vec{x} + e_i^{(k)} + \vec{y_j}) Z_j(\vec{x}_j, \vec{y_j}) \frac{g_i(\vec{x}_i + e_i^{(k)} + \vec{y_i})}{g_i(\vec{x}_i)} \\ &= \sum_{\vec{y_j}/|\vec{y_j}|>=0} \mu_i^{(k)} Z_j(\vec{x}_j, \vec{y_j}) \frac{g_i(\vec{x}_i + \vec{y_i})}{g_i(\vec{x}_i)} \rho_i^{(k)} \end{aligned}$$

The sum on $\vec{y_j}$ can be decomposed such that:

$$\Delta_{i,k} = \mu_i^{(k)} \rho_i^{(k)} \left[\sum_{\vec{y_i}/|\vec{y_i}|>0} Z_j(\vec{x}_j, \vec{y_j}) \frac{g_i(\vec{x}_i + \vec{y_i})}{g_i(\vec{x}_i)} + Z_j(\vec{x}_j, \vec{0}) \right]$$

Using lemma (3), we obtain:

$$\Delta_{i,k} = \mu_i^{(k)} \rho_i^{(k)} \left[Z_j(\vec{x}_j, \vec{0})(S_j - 1) + Z_j(\vec{x}_j, \vec{0}) \right]$$

After a simplification, we have:

$$\Delta_{i,k} = \mu_i^{(k)} \rho_i^{(k)} S_j Z_j(\vec{x}_j, \vec{0})$$

□

Lemma 5 *: For any type of service center, the following relation holds:*

$$\sum_{i=1}^{N} \sum_{k=1}^{C} \lambda_i^- N_{i,k}(\vec{x}_i) + \sum_{i=1}^{N} \lambda_i^- Z_i(\vec{x}_i, \vec{0}) = \sum_{i=1}^{N} \lambda_i^- \tag{5}$$

Proof: The proof is purely algebraic.

□

Proof of the theorem

Let us prove that the solution given by the theorem satisfies the Chapmann-Kolmogorov equation.

$$
\begin{aligned}
\Pi(\vec{x})\sum_{i=1}^{N}\sum_{k=1}^{C}\left[\lambda_i^{(k)}+M_{i,k}(\vec{x}_i)+\lambda_i^{-}N_{i,k}(\vec{x}_i)\right] = \\
& \sum_{i=1}^{N}\sum_{k=1}^{C}\lambda_i^{(k)}A_{i,k}(\vec{x}_i)\Pi(\vec{x}-e_i^{(k)}) \\
+ & \sum_{i=1}^{N}\sum_{k=1}^{C}M_{i,k}(\vec{x}+e_i^{(k)})d_i^{(k)}\Pi(\vec{x}+e_i^{(k)}) \\
+ & \sum_{i=1}^{N}\sum_{\vec{y_i}/|\vec{y_i}|>0}\lambda_i^{-}Z_i(\vec{x}_i,\vec{y_i})\Pi(\vec{x}+\vec{y_i}) \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{C}\sum_{l=1}^{C}M_{i,k}(\vec{x}+e_i^{(k)}-e_j^{(l)})A_{j,l}(\vec{x}_j)P_{ij}^{+(k,l)}\Pi(\vec{x}+e_i^{(k)}-e_j^{(l)}) \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\vec{y_j}/|\vec{y_j}|>=0}M_{i,k}(\vec{x}+e_i^{(k)}+\vec{y_j})Z_j(\vec{x}_j,\vec{y_j})P_{ij}^{-(k)}\Pi(\vec{x}+e_i^{(k)}+\vec{y_j})
\end{aligned}
$$

Dividing both sides by $\Pi(\vec{x})$ and using lemma 1 and 3, we have:

$$
\begin{aligned}
\sum_{i=1}^{N}\sum_{k=1}^{C}\left[\lambda_i^{(k)}+M_{i,k}(\vec{x}_i)+\lambda_i^{-}N_{i,k}(\vec{x}_i)\right] = \\
& \sum_{i=1}^{N}\sum_{k=1}^{C}\lambda_i^{(k)}A_{i,k}(\vec{x}_i)\frac{\Pi(\vec{x}-e_i^{(k)})}{\Pi(\vec{x})} && [1] \\
+ & \sum_{i=1}^{N}\lambda_i^{-}(S_i-1)Z_i(\vec{x}_i,\vec{0}) && [2] \\
+ & \sum_{i=1}^{N}\sum_{k=1}^{C}\mu_i^{(k)}d_i^{(k)}\rho_i^{(k)} && [3] \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{C}\sum_{l=1}^{C}\mu_i^{(k)}\rho_i^{(k)}A_{j,l}(\vec{x}_j)P_{ij}^{+(k,l)}\frac{\Pi(\vec{x}-e_j^{(l)})}{\Pi(\vec{x})} && [4] \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\vec{y_j}/|\vec{y_j}|>=0}\mu_i^{(k)}\rho_i^{(k)}S_jZ_j(\vec{x}_j,\vec{0})P_{ij}^{-(k)} && [5]
\end{aligned}
$$

We define now $\Lambda_{i,k}^{+}$ and Λ_i^{-} as follows:

$$
\left\{
\begin{array}{lll}
\Lambda_{i,k}^{+} & = & \lambda_i^{(k)}+\sum_{j=1}^{N}\sum_{l=1}^{C}\mu_j^{(l)}\rho_j^{(l)}P_{ji}^{+(l,k)} \\
\Lambda_i^{-} & = & \lambda_i^{-}+\sum_{j=1}^{N}\sum_{l=1}^{C}\mu_j^{(l)}\rho_j^{(l)}P_{ij}^{-(k)}
\end{array}
\right.
$$

Grouping term [1] with term [4] and term [2] with term [5], we obtain:

$$\sum_{i=1}^{N}\sum_{k=1}^{C}\left[\lambda_i^{(k)} + M_{i,k}(\vec{x}_i) + \lambda_i^- N_{i,k}(\vec{x}_i)\right] = \sum_{i=1}^{N}\sum_{k=1}^{C}\Lambda_{i,k}^+ A_{i,k}(\vec{x}_i)\frac{\Pi(\vec{x}-e_i^{(k)})}{\Pi(\vec{x})} + \sum_{i=1}^{N}\sum_{k=1}^{C} d_i^{(k)}\mu_i^{(k)}\rho_i^{(k)} + \sum_{i=1}^{N}\Lambda_i^- Z_i(\vec{x}_i,\vec{0})S_i - \sum_{i=1}^{N}\lambda_i^- Z_i(\vec{x}_i,\vec{0})$$

Using lemma 5, the Chapmann-Kolmogorov equation can be rewritten as follows:

$$\sum_{i=1}^{N}\sum_{k=1}^{C}\lambda_i^{(k)} + \sum_{i=1}^{N}\sum_{k=1}^{C} M_{i,k}(\vec{x}_i) + \sum_{i=1}^{N}\lambda_i^- = \sum_{i=1}^{N}\sum_{k=1}^{C}\Lambda_{i,k}^+ A_{i,k}(\vec{x}_i)\frac{\Pi(\vec{x}-e_i^{(k)})}{\Pi(\vec{x})} + \sum_{i=1}^{N}\sum_{k=1}^{C} d_i^{(k)}\mu_i^{(k)}\rho_i^{(k)} + \sum_{i=1}^{N}\Lambda_i^- Z_i(\vec{x}_i,\vec{0})S_i$$

Using the definition of $W_i(\vec{x}_i)$, we then have:

$$\sum_{i=1}^{N}\sum_{k=1}^{C}\left(\lambda_i^{(k)} + M_{i,k}(\vec{x}_i)\right) + \sum_{i=1}^{N}\left(\lambda_i^- + \Lambda_i^- S_i W_i(\vec{x}_i)\right) = \sum_{i=1}^{N}\sum_{k=1}^{C}\Lambda_{i,k}^+ A_{i,k}(\vec{x}_i)\frac{\Pi(\vec{x}-e_i^{(k)})}{\Pi(\vec{x})} + \sum_{i=1}^{N}\sum_{k=1}^{C} d_i^{(k)}\mu_i^{(k)}\rho_i^{(k)} + \sum_{i=1}^{N}\Lambda_i^- S_i$$

The previous system can be decomposed into 2 equations, a flow equation and a state-dependent equation:

$$\sum_{i=1}^{N}\sum_{k=1}^{C}\lambda_i^{(k)} + \sum_{i=1}^{N}\lambda_i^- = \sum_{i=1}^{N}\sum_{k=1}^{C}\mu_i^{(k)}\rho_i^{(k)}d_i^{(k)} + \sum_{i=1}^{N}\Lambda_i^- S_i \tag{6}$$

$$\sum_{i=1}^{N}\sum_{k=1}^{C}\Lambda_{i,k}^+ A_{i,k}(\vec{x}_i)\frac{\Pi(\vec{x}-e_i^{(k)})}{\Pi(\vec{x})} = \sum_{i=1}^{N}\sum_{k=1}^{C} M_{i,k}(\vec{x}_i) + \sum_{i=1}^{N}\Lambda_i^- S_i W_i(\vec{x}_i) \tag{7}$$

Let's first prove that equation (6) is a flow equation. For that, consider the assumption described by equation (1). Equation (6) can then be rewritten as follows:

$$\sum_{i=1}^{N}\sum_{k=1}^{C}\Lambda_{i,k}^{+}+\sum_{i=1}^{N}\lambda_i^{-}=\sum_{i=1}^{N}\sum_{k=1}^{C}\mu_i^{(k)}\rho_i^{(k)}\left(1-\sum_{j=1}^{N}P_{ij}^{-(k)}\right)+\sum_{i=1}^{N}\Lambda_i^{-}S_i$$

After some algebraic manipulations, we obtain:

$$\sum_{i=1}^{N}\sum_{k=1}^{C}\Lambda_{i,k}^{+}=\sum_{i=1}^{N}\sum_{k=1}^{C}\mu_i^{(k)}\rho_i^{(k)}+\sum_{i=1}^{N}\Lambda_i^{-}(S_i-1)$$

As $S_i-1=S_i\sum_{k=1}^{C}\rho_i^{(k)}p_i^{(k)}$, we then have:

$$\sum_{i=1}^{N}\sum_{k=1}^{C}\Lambda_{i,k}^{+}=\sum_{i=1}^{N}\sum_{k=1}^{C}\rho_i^{(k)}\left[\mu_i^{(k)}+\Lambda_i^{-}S_ip_i^{(k)}\right]$$

This equation is equal to system (2) summed on i and k. Thus the proof is complete.

□

Now, let's prove that equality (7) holds for each service station type.

$$\sum_{i=1}^{N}\sum_{k=1}^{C}\Lambda_{i,k}^{+}A_{i,k}(\vec{x}_i)\frac{\Pi(\vec{x}-e_i^{(k)})}{\Pi(\vec{x})}=\sum_{i=1}^{N}\sum_{k=1}^{C}M_{i,k}(\vec{x}_i)+\sum_{i=1}^{N}\Lambda_i^{-}S_iW_i(\vec{x}_i)$$

For each type of service center, we replace $M_{i,k}(\vec{x}_i)$, $W_i(\vec{x}_i)$ and $A_{i,k}(\vec{x}_i)$ by their expressions and we obtain:

- FIFO:

$$\sum_{k=1}^{C}\Lambda_{i,k}^{+}\frac{\Pi(\vec{x}-e_i^{(k)})}{\Pi(\vec{x})}1\!\!1_{\{r_{i,\infty}=k\}}=\sum_{k=1}^{C}\mu_i^{(k)}1\!\!1_{\{r_{i,1}=k\}}+\sum_{k=1}^{C}\Lambda_i^{-}S_ip_i^{(k)}1\!\!1_{\{r_{i,1}=k\}}$$

After some algebraic manipulations and using the expression of $\rho_i^{(k)}$, we then get:

$$\sum_{k=1}^{C}\left(\mu_i^{(k)}+\Lambda_i^{-}S_ip_i^{(k)}\right)1\!\!1_{\{r_{i,\infty}=k\}}=\sum_{k=1}^{C}\mu_i^{(k)}1\!\!1_{\{r_{i,1}=k\}}+\sum_{k=1}^{C}\Lambda_i^{-}S_ip_i^{(k)}1\!\!1_{\{r_{i,1}=k\}}$$

which leads to:

$$\sum_{k=1}^{C}\left(\mu_i^{(k)}+\Lambda_i^{-}S_ip_i^{(k)}\right)1\!\!1_{\{r_{i,\infty}=k\}}=\sum_{k=1}^{C}\left(\mu_i^{(k)}+\Lambda_i^{-}S_ip_i^{(k)}\right)1\!\!1_{\{r_{i,1}=k\}}$$

To be satisfied, the equality imposes the following constraints:

$$\begin{cases}\mu_i^{(k)} & = & \mu_i\\ p_i^{(k)} & = & p_i\end{cases}$$

□

- LIFO:

$$\sum_{k=1}^{C} \Lambda_{i,k}^{+} \frac{\Pi(\vec{x} - e_i^{(k)})}{\Pi(\vec{x})} \mathbb{1}_{\{r_{i,1}=k\}} = \sum_{k=1}^{C} \mu_i^{(k)} \mathbb{1}_{\{r_{i,1}=k\}} + \sum_{k=1}^{C} \Lambda_i^{-} S_i p_i^{(k)} \mathbb{1}_{\{r_{i,1}=k\}}$$

This leads to the following equality:

$$\sum_{k=1}^{C} \left(\mu_i^{(k)} + \Lambda_i^{-} S_i p_i^{(k)}\right) \mathbb{1}_{\{r_{i,1}=k\}} = \sum_{k=1}^{C} \mu_i^{(k)} \mathbb{1}_{\{r_{i,1}=k\}} + \sum_{k=1}^{C} \Lambda_i^{-} S_i p_i^{(k)} \mathbb{1}_{\{r_{i,1}=k\}}$$

Obviously, the left hand side and right hand side are the same.

$$\sum_{k=1}^{C} \left(\mu_i^{(k)} + \Lambda_i^{-} S_i p_i^{(k)}\right) \mathbb{1}_{\{r_{i,1}=k\}} = \sum_{k=1}^{C} \left(\mu_i^{(k)} + \Lambda_i^{-} S_i p_i^{(k)}\right) \mathbb{1}_{\{r_{i,1}=k\}}$$

This completes the proof.

□

- PS:

$$\sum_{k=1}^{C} \Lambda_{i,k}^{+} \frac{\Pi(\vec{x} - e_i^{(k)})}{\Pi(\vec{x})} \mathbb{1}_{\{x_i^{(k)}>0\}} = \sum_{k=1}^{C} \mu_i^{(k)} \frac{x_i^k}{|\vec{x}_i|} \mathbb{1}_{\{|\vec{x}_i|>0\}} + \sum_{k=1}^{C} \Lambda_i^{-} S_i p_i^{(k)} \frac{x_i^k}{|\vec{x}_i|} \mathbb{1}_{\{|\vec{x}_i|>0\}}$$

Using equation (2), we then have:

$$\sum_{k=1}^{C} \left(\mu_i^{(k)} + \Lambda_i^{-} S_i p_i^{(k)}\right) \frac{x_i^k}{|\vec{x}_i|} \mathbb{1}_{\{x_i^{(k)}>0\}} = \sum_{k=1}^{C} \mu_i^{(k)} \frac{x_i^k}{|\vec{x}_i|} \mathbb{1}_{\{|\vec{x}_i|>0\}} + \sum_{k=1}^{C} \Lambda_i^{-} S_i p_i^{(k)} \frac{x_i^k}{|\vec{x}_i|} \mathbb{1}_{\{|\vec{x}_i|>0\}}$$

Since $x_i^k > 0$ implies that $|\vec{x}_i| > 0$, this equation can then be rewritten as follows:

$$\sum_{k=1}^{C} \left(\mu_i^{(k)} + \Lambda_i^{-} S_i p_i^{(k)}\right) \frac{x_i^k}{|\vec{x}_i|} \mathbb{1}_{\{|\vec{x}_i|>0\}} = \sum_{k=1}^{C} \left(\mu_i^{(k)} + \Lambda_i^{-} S_i p_i^{(k)}\right) \frac{x_i^k}{|\vec{x}_i|} \mathbb{1}_{\{|\vec{x}_i|>0\}}$$

This completes the proof.

□