Multiple Class G-Networks with List Oriented Deletions

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Abstract

We study a new dynamic of generalized networks of queues. We consider networks with multiple classes of customers and signals, and three types of service disciplines: FIFO, LIFO/PR and Processor Sharing. Moreover, we consider a probabilistic deleting mechanism based on iterated deletions of customers according to an arbitrary list of service stations. We prove that these networks have a steady-state product form solution and we investigate the existence of a solution to the traffic equation of such networks.

1 Introduction

Generalized networks (G-networks for short) is one of the newest topics in queueing networks theory. Since they have been introduced by Gelenbe in [12][17], they have received considerable attention as, despite synchronized transitions between customers, they still exhibit product-form steady-state distribution. As synchronizations between queues (for instance synchronized arrivals generated by a fork) usually make the model non-product form, G-networks appear as a very appealing model to represent and analyze systems with synchronized transitions. However, it must be very clear that complex synchronizations that we obtain with G-networks are not necessarily equivalent to synchronizations we already use in other modeling techniques such as Petri nets.

G-networks synchronizations are based on the interaction between customers and signals. In the early papers, signals were also denoted as negative customers. They are sent into a queue, they may act upon some customers present in the queue and disappear instantaneously. Signals are not queued in the network and never receive service.

As customers may, at the completion of their service, become signals and be routed into another queue, G-networks exhibit some synchronized transitions which are not modeled by Jackson networks. Various effects of signals have been investigated to prove a product-form solution. But they quite always imply that the customer in service leaves the queue immediately. So the synchronizations described by G-networks contain the simultaneous departures of some customers from two queues.

These networks have a steady-state product form solution under usual Markovian assumptions: Poisson arrivals of external customers and signals, i.i.d. exponential service times and probabilistic routing. We also assume that the queues have an infinite capacity.

Basic interactions between customers and signals have been considered in [13][16][18]. In this paper, we prove that product form solution is preserved even if the G-networks have a very complex dynamic based on the iterated deletions of customers from an arbitrary list of stations. We prove these results for networks with multiple classes of customers and multiple classes of signals. Three service disciplines are considered: FIFO, LIFO and PS. The deletion probabilities at each step of the iterated mechanism may depend upon the classes of the signal and the customer but also upon the step of the iteration. We also show that such a dynamic may be obtained, in an heuristic way, as the limiting behavior of some basic synchronizations which are described by Gelenbe as G-networks with triggers [14] in a monoclass paradigm and which has been recently extended by Gelenbe et al. [19] for multiclass networks. So, we emphasize Gelenbe's result on networks with triggers as an heuristic to design more complex networks with product form solution.

Our model is a generalization of former results on multiclass G-networks with exponential service times and where the signals delete a customer or a group of customers. These are described below. For the sake of completeness, one may add the results of Chao [2]. His approach is quite different as he assumes that the service time distribution is Coxian and that the signals only trigger the end of a phase of the service.

- In [7], Fourneau et al. have presented the first G-networks with multiple classes of positive and negative customers. Negative customers may be viewed as signals which delete only the positive customer in service. Three types of service disciplines were considered: FIFO, LIFO, and PS. The service times are class dependent except for FIFO queue. And the probability that a negative customer succeeds in deleting a positive customer is dependent on the classes of the two customers involved in the interaction. Thus, multiple classes of negative customers allow to represent various deletion abilities. As usual, there exist some restrictions on the deletion probability for FIFO queues.

- In [19], authors have generalized Gelenbe's results on networks with triggers using the same approach as in [7]. The limitations are the same.

- In [5], authors have presented a network of PS queues where the signals flush the queue. Then, they have generalized to more complex networks of queues with FIFO, LIFO and PS disciplines and iterated deletions [6]. This last result is related to our paper. In [6], the authors consider that the deletion process is iterative but it is assumed that the iterations remain on the same queue. Thus, a signal sent by a station synchronizes only two queues. It is denoted as a $(-1, -a)$ synchronization as a customer at the completion of its service leaves the queue and is changed into a signal which deletes a random number a of customers. The distribution of a and its decomposition in classes depend on the state of the queue, the service discipline and the probabilities of success at each step of the iterative deletion mechanism.

Our model is much more complex since we assume that the deletion process takes place in an arbitrary list of queues. It is also related to new results presented by Serfozo and Yang in [21]. They proved that the list-oriented deletion process (denoted as string) have a product form solution. Briefly, their model is a single class model and does not include a description of service discipline for multiclass networks. They only suggest in some examples to apply their results to networks with PS service discipline.

The paper is organized as follows. In section 2 we present an example of a complex dynamic with product form, we also show how this behavior is modeled by a G-network with triggers and instantaneous services. We present our model in section 3 and in section 4, we prove that it has a product form solution. In section 5, we present some examples and we also show that some of the previous results already published are corollaries of our results. In this section, is also discussed the existence of a solution to the flow equation using a new kind of arguments based on equivalence between networks.

2 Complex behaviors as asymptotic of G-networks with triggers

We recall the basic model of G-networks with triggers with a single class of customers. For the sake of simplicity, we restrict ourselves to single class networks. This section is mainly concerned with the asymptotic behavior of these networks when some service rates grow to infinity (i.e. the service-time distribution for some stations will become smaller and smaller).

2.1 G-networks with triggers

Gelenbe considers a network with N stations. The service times are independent and exponentially distributed with parameter μ_i. Customers (resp. signals) arrive from the outside according to independent Poisson processes with rate λ_i^+ (resp. λ_i^-). The routing is Markovian. At the completion of its service in station i, a customer is routed to station j with probability P_{ij}^+, goes outside with probability d_i or becomes a signal with probability P_{ij}^-. When a signal enters station j, it triggers a customer to station k with probability Q_{jk} or to the outside with probability D_j. Thus G-networks with triggers exhibit synchronization between three queues: a departure at the service completion time in queue i, the triggered movement of a customer in queue j, the arrival of this customer in queue k. Let n_i be the number of customers in station i and $\vec{n}$ the network state in equilibrium, state transition rates are:

$$\begin{array}{rcl} \mathcal{Q}_{\vec{n},\vec{n}-\vec{e_i}-\vec{e_j}+e_k} & = & \mu_i P_{ij}^- Q_{jk} \mathbb{1}_{\{n_i>0\}} \mathbb{1}_{\{n_j>0\}} \\ \mathcal{Q}_{\vec{n},\vec{n}-\vec{e_i}-\vec{e_j}} & = & \mu_i P_{ij}^- D_j \mathbb{1}_{\{n_i>0\}} \mathbb{1}_{\{n_j>0\}} \\ \mathcal{Q}_{\vec{n},\vec{n}-\vec{e_i}+\vec{e_j}} & = & \left(\mu_i P_{ij}^+ + \lambda_i^- Q_{ij}\right) \mathbb{1}_{\{n_i>0\}} \\ \mathcal{Q}_{\vec{n},\vec{n}-\vec{e_i}} & = & \left(\mu_i d_i + \lambda_i^- D_i + \mu_i P_{ij}^- \mathbb{1}_{\{n_j=0\}}\right) \mathbb{1}_{\{n_i>0\}} \\ \mathcal{Q}_{\vec{n},\vec{n}+\vec{e_i}} & = & \lambda_i^+ \end{array} \tag{1}$$

Let us now present Gelenbe's theorem for G-networks with triggers (see [14] for a detailed proof) :

Theorem 1 *(Gelenbe) Consider an open G-network with triggers, if the non linear flow equation*

$$\rho_i = \frac{\lambda_i^+ + \sum_j \mu_j \rho_j P_{ji}^+ + \sum_j \sum_k \mu_j \rho_j P_{jk}^- \rho_k Q_{ki}}{\mu_i + \sum_j \mu_j \rho_j P_{ji}^-} \tag{2}$$

has a solution such that

$$\rho_i < 1 \quad \forall i \in [1..N] \tag{3}$$

then the steady-state distribution exists and has a product form

$$p(\vec{n}) = \prod_{i=1}^{N} (1-\rho_i)\rho_i^{n_i} \tag{4}$$

2.2 Examples

Consider the G-network with triggers depicted in figure 1. We assume that queue 1 may send a signal to queue 2 with probability P_{12}^-. When a signal arrives in queue 2, it moves a customer to queue 4. After a service in queue 4 the customer is sent into queue 3 as a signal with probability 1. In that queue, the signal triggers a customer from queue 3 to queue 5. In queue 5, the customer receives service and is routed as signal into queue 2. The routing of customers and signals is completely specified by the three routing matrices associated to this network:

$$P^+ = \begin{pmatrix} 0 & P_{12}^+ & P_{13}^+ & 0 & 0 \\ P_{21}^+ & 0 & P_{23}^+ & 0 & 0 \\ P_{31}^+ & P_{32}^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad P^- = \begin{pmatrix} 0 & P_{12}^- & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (5)$$

The external customers arrive according to Poisson processes with rate λ_i^+ for station i, $i = 1, 2$ and 3. Stations 4 and 5 do not receive customers from the outside. We also assume that there is no external signals arriving in the network. Let μ_4 and μ_5 be the service rates in stations 4 and 5. Remember that the service times have an exponential distribution and the steady-state distribution has a product form for all values of μ_4 and μ_5.

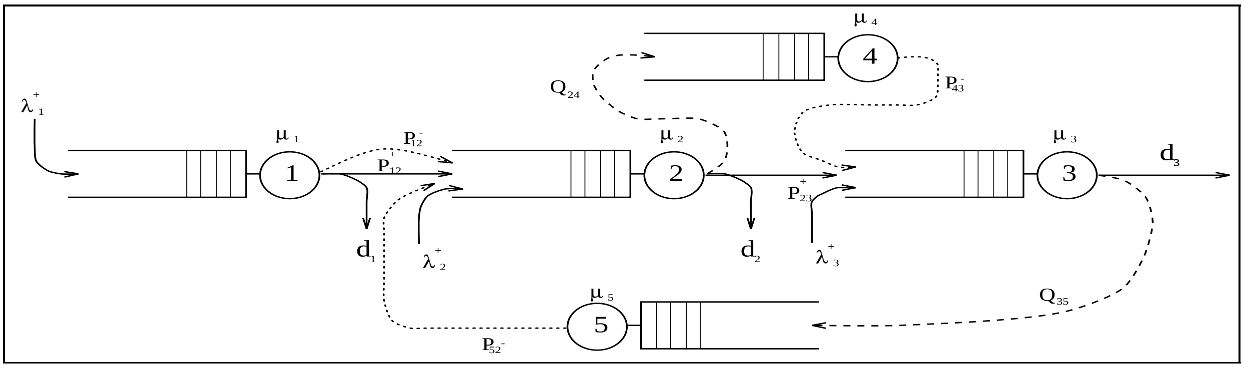

Figure 1: G-network with triggers

2.2.1 G-networks with synchronized partial flushing

Now, let us consider a new interaction between signals and customers which define a new type of G-networks. This interaction is depicted in figure 2 to emphasis the relation with the G-network with triggers illustrated in figure 1. This network contains three queues. The services are exponential with rate μ_i for station i. The customers move between stations according to a routing matrix $3 \times 3 : P^+$. We assume that the external arrivals in queue i follow independent Poisson processes with the same rate λ_i^+ as in the previous model. Queue 1 sends signals to queue 2. Let $n_2(t)$ and $n_3(t)$ be the number of customers in respectively queues 2 and 3 just before the arrival of a signal.

A signal at its arrival into queue 2, acts as follows:

$$\begin{cases} \text{If } \; n_2(t) \leq n_3(t) \; \text{ then } & n_2(t+\varepsilon) = 0 \text{ and } n_3(t+\varepsilon) = n_3(t) - n_2(t) \\ \text{If } \; n_2(t) > n_3(t) \; \text{ then } & n_2(t+\varepsilon) = n_2(t) - n_3(t) - 1 \text{ and } n_3(t+\varepsilon) = 0 \end{cases} \tag{6}$$

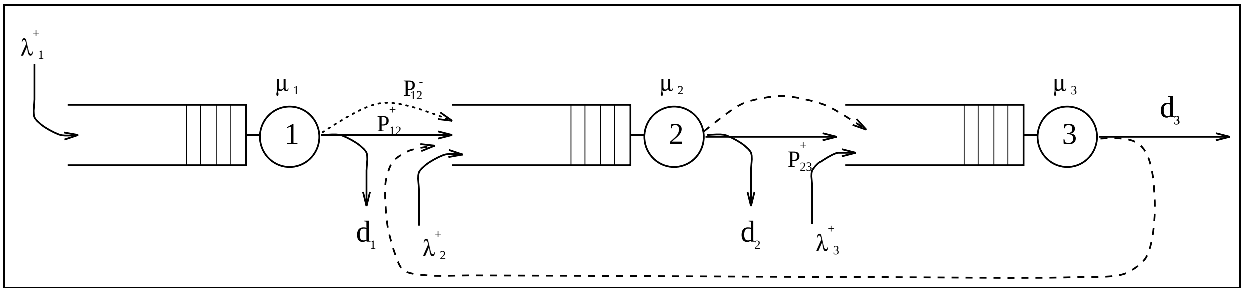

Figure 2: Associated network with synchronized partial flushing

This network is a simple example of G-networks with synchronized partial flushing which has been defined in [22]. It has been proved in [8] that multiclass G-networks with synchronized partial flushing have a product-form steady-state distribution.

2.2.2 Increasing the service-rates of triggered customers

Clearly, both previous models are related. Consider the G-network with triggers in figure 1. There is a directed cycle consisting of stations $(2, 4, 3, 5)$. At each time the cycle is completed, two customers disappear from the network: one is moved from station 2 to station 4 where it receives service and becomes a signal which disappears when it arrives in station 3. The other one disappears after transitions between queues 3, 5 and 2. The cycle can be used until a signal arrives into an empty queue. The completion time of a cycle is the sum of the two sojourn times in stations 4 and 5 as the effect of signals in stations 2 and 3 is instantaneous.

Assume now that we have modified the service-time distribution in queues 4 and 5 to be instantaneous. We do not claim, at this moment, that this new network has a product form solution. The idea is just to compare the dynamics of this network with the dynamics of the G-network with synchronized partial flushing depicted in figure 2. To the best of our knowledge, queueing networks with instantaneous services have not yet been considered, mainly because Jackson's networks with instantaneous services still have the same dynamics.

Consider the effect of a signal which enters queue 2 arriving from queue 1. Assume that there are some customers in queues 2 and 3 at that time. The signal triggers a customer into queue 4 where it is transformed into a signal which joins queue 3 instantaneously. The second part of the cycle is also performed instantaneously: the signal moves up a customer from queue 3 to queue 5 where it is transformed into a signal which enters queue 2. Clearly, this dynamic will continue to delete customers from queues 2 and 3 until it vanishes because it enters an empty queue. So the effect of this dynamic is to suppress in an iterative manner one customer in queue 2 and one customer in queue 3 until queue 2 or queue 3 is empty. Clearly, the number of customers deleted by the signal is exactly defined by the dynamics of the G-network with synchronized partial flushing (figure 2).

But this network may be approximated as close as we need by the G-network with triggers of figure 1. Remember that this network has a product-form solution for all values of service rates μ_4 and μ_5 as soon as the flow equation has a solution smaller than 1. Clearly increasing the service rate makes the system more stable and the flow equation still has a solution if we increase the service rates from a point where the solution exists. Assume now that one increases the service rates (i.e. in station 4 and 5) continuously without any modification to the other parameters. If μ_4 and μ_5 grow to infinity, then the service times and the sojourn times in queues 4 and 5 tends to zero; the services in these stations become close to instantaneous. So we may approximate the unknown solution of network of figure 2 by the product-form solution of the network depicted in figure 1. This paper is motivated by such an idea. We do not prove that the limits exist but we consider all the dynamics which may be defined by G-networks with triggers and instantaneous services and we prove using global balance equation that these networks have product form solution.

3 The Model

We consider a model of N queues, C classes of customers and H classes of signals. As in BCMP theorem [1], we consider that the service discipline in the queues can be of different types. We will refer to these discipline types using the same type numbers as in [1] and the one used by Gelenbe in [7]. The discipline types we use are the following:

- Type 1: first-in-first-out (FIFO),
- Type 2: processor sharing (PS),
- Type 4: last-in-last-out with preemptive resume priority (LIFO/PR).

Service centers with an infinite number of servers are not considered here since this type of discipline does not fit with our model. This is the type 3 described in [1].

We consider queues where external arrivals follow independent Poisson processes. We denote by $\lambda_i^{(k)}$ the external arrival rate of class k customers to queue i. Note that without loss of generality, we do not consider external arrivals of signals.

All services are exponentially distributed, but only customers receive service. For customers of class k in queue i, the service rate is noted $\mu_{i,k}$. At its service completion in station i, a class k customer may either leave the network with probability $d_i^{(k)}$, or be routed to another service center j. In this case, the customer is routed either as a customer of class m with probability $P_{ij}^{+(k,m)}$ or as a signal of type h with probability $Q_{i,j}^{k,h}$. Therefore, we have to set the following relation:

$$\forall i, k \quad \sum_{j=1}^{N} \sum_{m=1}^{C} P_{ij}^{+(k,m)} + \sum_{j=1}^{N} \sum_{h=1}^{H} Q_{i,j}^{k,h} + d_i^{(k)} = 1 \tag{7}$$

To each signal of type h corresponds a list S_h of queues the signal has to visit in the order they appear in that list. The role of the signal is to remove one customer from each queue it visits. The list length may be finite or infinite, we note L_h its size.

So, when a signal of type h arrives to queue j, it tries to remove a customer with a certain probability and according to the service discipline of that queue. The removing probability will depend on the type of the signal, the queue it visits, the customer class it tries to remove and the position x of the station in the list S_h. We will note this probability $D_{j,m,h,x}$.

Note that a queue j may be visited several times by the signal because it may appear several times in the list. At each visit to this queue, it may also remove a customer of the same class m. We denote by $\eta(h, L_h, j, m)$ the number of occurrences of couple (j, m) in the list of length L_h associated to signal h.

If in one of the stations visited the signal fails in removing a customer, then it vanishes and the deleting process stops. In revenge, if the signal succeeds the removal in each queue it visits, it has then to add with a certain probability a new customer to a queue. This queue is the last one the signal visits before it vanishes and does not appear in the list S_h. We will say that this queue is the $(L_h + 1)$th queue of the list. The customer added in this queue can be considered as a customer resulting from the L_h customers removed by the signal from the different queues. We note $\alpha_{L_h+1,l}$ the probability to succeed in adding a new customer of class l in queue $L_h + 1$.

Note that in the case of infinite list, the new customer can not be created because the deleting process will be interrupted by the visit of an empty queue. So, the signal will immediately vanish.

3.1 State Representation

The state of the network will be denoted by the vector $\vec{n} = (\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N)$ where component $\vec{n}_i$ is the state of service station i. For all service disciplines, the number of customers in a station i will be denoted by $|\vec{n}_i|$.

The state of each station will depend on the service type of this station. For a FIFO or LIFO service station i, this state will be described by the vector $\vec{n}_i = (r_{i,1}, r_{i,2}, ..., r_{i,\infty})$ whose length is variable and whose component $r_{i,x}$ denotes the class of the xth customer in the queue. In particular, $r_{i,\infty}$ denotes the class of the last customer in queue i. Note that the first customer in a FIFO or LIFO station is the one in service.

For a PS service center, the state is represented by vector $\vec{n}_i = (n_i^1, n_i^2, \ldots, n_i^C)$ whose component n_i^k is the number of customers of class k in the queue i.

3.2 Notations

We denote by $(\vec{n}_i + e_{i,k})$ the state of queue i with one more customer of class k. Note that this customer is the one in service for FIFO and LIFO service centers. In PS, it adds one to the kth component. The notation $(\vec{n}_i - e_{i,k})$ is used to describe the converse situation.

As $v(h, x)$ denotes the service station number visited by the signal of type h at the xth position in the list S_h, $z(x)$ denotes the customer class removed from this station during this visit. In particular, $v(h, L_h + 1)$ refers to the service center visited at the $L_h + 1$ position, which means the queue which receives the new customer.

Besides S_h which is an ordered list of service stations visited by a signal of type h, we define $\vec{y}$ as a vector which corresponds to an ordered list of removed customer classes. This vector is not unique and will take its values from the set of all possible lists of customer classes that may be associated to signal h.

Since the removal process may be, as explained before, interrupted for different reasons, we decompose this set into 2 subsets, $\Gamma(h)$, a set of all possible lists of customer classes whose length is L_h and $\Delta(h)$, a set of all possible lists of customer classes whose length is less than L_h. If $\vec{y}$ components are elements of a list belonging to set $\Gamma(h)$ then $|\vec{y}| = L_h$. If $\vec{y}$ components are elements of a list belonging to set $\Delta(h)$ then $|\vec{y}| < L_h$.

Note that we assume without loss of generality that to all service stations arrive customers of all classes either from outside the network or from other service stations in the network.

To have a better idea of the subsets $\Gamma(h)$ and $\Delta(h)$ contents, consider the G-network with synchronized partial flushing depicted in figure 2. Assume that signals of this network may be of only 2 types, h_1 and h_2, and that there are only 2 classes of customers, c_1 and c_2.

- To a signal of type h_1 we associate the ordered list $S_{h_1} = \{2,3\}$. So, this list contains the station numbers the signal of type h_1 has to visit.

  The subset $\Gamma(h_1)$ which is defined as the set of all possible lists of customer classes whose length is $L_{h_1} = 2$ will contain 4 lists of classes as follows:

$$\Gamma(h_1) = \{\{c_1,c_1\},\{c_1,c_2\},\{c_2,c_1\},\{c_2,c_2\}\}$$

- To a signal of type h_2 we associate the ordered list $S_{h_2} = \{2,3,2,3,2,3,\ldots\}$. This list is infinite, so $L_{h_2} = \infty$ and subset $\Gamma(h_2)$ will be as follows:

$$\begin{aligned}\Gamma(h_2) \quad = \quad & \{\{c_1,c_1,c_1,\ldots,c_1,\ldots\},\{c_2,c_1,c_1,\ldots,c_1,\ldots\},\{c_1,c_2,c_1,\ldots,c_1,\ldots\},\\ & \{c_1,c_1,c_2,\ldots,c_1,\ldots\},\ldots,\{c_1,c_1,c_1,\ldots,c_2,\ldots\},\ldots,\\ & \{c_2,c_2,c_1,\ldots,c_1,\ldots\},\{c_2,c_1,c_2,\ldots,c_1,\ldots\},\ldots,\\ & \{c_2,c_1,c_1,\ldots,c_2,\ldots\},\ldots,\{c_2,c_2,c_2,\ldots,c_2,\ldots\}\}\end{aligned}$$

Now and before we give the main theorem and its proof, we need to introduce some notations to carry out algebraic manipulations with some independence from the service center type.

- The state-dependent service rates for customers of class k at service station i will be denoted by $M_{i,k}(\vec{n}_i)$ where $\vec{n}_i$ refers to the state of the service station i. From the definition of the service rates $\mu_{i,k}$ and according to the service center type, we have:
  - FIFO/LIFO: $M_{i,k}(\vec{n}_i) = \mu_{i,k}\mathbb{1}_{\{r_{i,1}=k\}}$
  - PS: $M_{i,k}(\vec{n}_i) = \mu_{i,k}\frac{n_i^k}{|\vec{n}_i|}\mathbb{1}_{\{|\vec{n}_i|>0\}}$
- We denote by $A_{i,k}(x_i)$ the condition which shows that it is possible to be in state $\vec{n}_i$ after the arrival of a positive customer of class k:
  - FIFO: $A_{i,k}(\vec{n}_i) = \mathbb{1}_{\{r_{i,\infty}=k\}}$
  - LIFO: $A_{i,k}(\vec{n}_i) = \mathbb{1}_{\{r_{i,1}=k\}}$

- PS: $A_{i,k}(\vec{n}_i) = 1\!\!1_{\{n_i^k>0\}}$

- N_i denotes the conditions, given the system in state $\vec{n}_i$, that an incoming signal of type h does not succeed in deleting any customer in queue i:
  - FIFO: $N_{i,k}(\vec{n}_i) = 1\!\!1_{\{|\vec{n}_i|=0\}} + \sum_{k=1}^{C}(1 - D_{i,k,h,x})1\!\!1_{\{r_{i,1}=k\}}$ where $D_{i,k,h,x} = D_i \; \forall k, h, x$
  - LIFO: $N_{i,k}(\vec{n}_i) = 1\!\!1_{\{|\vec{n}_i|=0\}} + \sum_{k=1}^{C}(1 - D_{i,k,h,x})1\!\!1_{\{r_{i,1}=k\}}$
  - PS: $N_{i,k}(\vec{n}_i) = 1\!\!1_{\{|\vec{n}_i|=0\}} + \sum_{k=1}^{C}(1 - D_{i,k,h,x})\frac{n_i^k}{|\vec{n}_i|}1\!\!1_{\{|\vec{n}_i|>0\}}$

- $Z(h, \vec{n}, \vec{u})$ denotes the probability to be in state $\vec{n}$ after the arrival of a signal of type h which destroys $|\vec{u}|$ customers:
  - FIFO/LIFO:
    $Z(h, \vec{n}, e_{i,k}) = D_{i,k,h,1}$
    $Z(h, \vec{n}, \vec{u} + e_{i,k}) = Z(h, \vec{n}, \vec{u})D_{i,k,h,|\vec{u}|+1}$
  - PS:
    $Z(h, \vec{n}, e_{i,k}) = \frac{n_i^k+1}{|n_i|+1} \; D_{i,k,h,1}$
    $Z(h, \vec{n}, \vec{u} + e_{i,k}) = Z(h, \vec{n}, \vec{u}) \; \frac{n_i^k+\eta(h,L_h,i,k)+1}{|n_i|+\eta(h,L_h,i)+1} \; D_{i,k,h,|\vec{u}|+1}$

4 Product Form Theorem

Let $\Pi(\vec{n})$ be the stationary probability distribution of the network state if it exists. The following theorem establishes the existence of a product form solution of the network type being considered.

Theorem 2 : *Consider an arbitrary open G-network with C classes of positive customers and H classes of signals. If the system of non-linear equations:*

$$\rho_{i,k} = \frac{\lambda_i^{(k)} + \sum_{j=1}^{N}\sum_{m=1}^{C}\mu_{j,m}\rho_{j,m}P_{ji}^{+(m,k)} + \Lambda_{i,k}^{+}}{\mu_{i,k} + \Lambda_{i,k}^{-}} \tag{8}$$

where

$$\begin{cases} \Lambda_{i,k}^{+} = \displaystyle\sum_{j=1}^{N}\sum_{h=1}^{H}\sum_{\vec{y}\in\Gamma(h)} \mu_{j,m}\rho_{j,m}Q_{j,i}^{m,h}\prod_{x=1}^{L_h}\left(\rho_{v(h,x),z(x)}D_{v(h,x),z(x),h,x}\right)\alpha_{i,k}1\!\!1_{\{L_h<\infty\}}1\!\!1_{\{v(h,L_h+1)=i\}} \\ \Lambda_{i,k}^{-} = \displaystyle\sum_{j=1}^{N}\sum_{m=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Lambda(h)} \mu_{j,m}\rho_{j,m}Q_{j,i}^{m,h}\prod_{x=1}^{|\vec{y}|}\left(\rho_{v(h,x),z(x)}D_{v(h,x),z(x),h,x}\right)D_{i,k,h,|\vec{y}|+1}1\!\!1_{\{v(h,|\vec{y}|+1)=i\}} \end{cases}$$

has a positive solution such that $\forall \;\; i, k \;\; \rho_{i,k} > 0$ and for each service station i, $\sum_{k=1}^{C}\rho_{i,k} < 1$ then the stationary distribution of the network state exists and has a product form:

$$\Pi(\vec{n}) = G \prod_{i=1}^{N} g_i(\vec{n}_i) \tag{9}$$

where $g_i(n_i)$ depends on the type of the service station i as follows:

- *FIFO/LIFO:*

$$g_i(\vec{n}_i) = \prod_{n=1}^{|\vec{n}_i|} \rho_i^{r_{i,n}}$$

- *PS:*

$$g_i(\vec{n}_i) = |\vec{n}_i|! \prod_{k=1}^{C} \frac{(\rho_i^k)^{n_i^k}}{n_i^k!}$$

and G is the normalization constant.

Proof of theorem2: The proof consists mainly of algebraic manipulations of the following Chapmann-Kolmogorov equation:

$$\begin{aligned}
\Pi(\vec{n}) \sum_{i=1}^{N}\sum_{k=1}^{C} \left[\lambda_i^{(k)} + M_{i,k}(\vec{n}_i)\right] = &\\
& \sum_{i=1}^{N}\sum_{k=1}^{C} \lambda_i^{(k)} A_{i,k}(\vec{n}_i)\Pi(\vec{n} - e_{i,k}) \\
+ & \sum_{i=1}^{N}\sum_{k=1}^{C} M_{i,k}(\vec{n} + e_{i,k}) d_i^{(k)} \Pi(\vec{n} + e_{i,k}) \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{C}\sum_{m=1}^{C} M_{i,k}(\vec{n} + e_{i,k} - e_{j,m}) A_{j,m}(\vec{n}_j) P_{ij}^{+(k,m)} \Pi(\vec{n} + e_{i,k} - e_{j,m}) \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{C}\sum_{m=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Gamma(h)} M_{i,k}(\vec{n} + e_{i,k} + e_{L_h} - e_{j,m}) A_{j,m}(\vec{n}_j) Q_{i,j}^{k,h} \Pi(\vec{n} + e_{i,k} + e_{L_h} - e_{j,m}) \\
& \qquad Z(h,\vec{n},\vec{y}) \alpha_{j,m} \mathbb{1}_{\{v(h,L_h+1)=j\}} \mathbb{1}_{\{L_h<\infty\}} \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{C}\sum_{m=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Gamma(h)} M_{i,k}(\vec{n} + e_{i,k} + e_{L_h}) Q_{i,j}^{k,h} \Pi(\vec{n} + e_{i,k} + e_{L_h}) \\
& \qquad Z(h,\vec{n},\vec{y})(1 - \alpha_{j,m}) \mathbb{1}_{\{v(h,L_h+1)=j\}} \mathbb{1}_{\{L_h<\infty\}} \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{C}\sum_{m=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Lambda(h)} M_{i,k}(\vec{n} + e_{i,k} + e_{|\vec{y}|}) Q_{i,j}^{k,h} \Pi(\vec{n} + e_{i,k} + e_{|\vec{y}|}) \\
& \qquad Z(h,\vec{n},\vec{y}) N_{j,m}(\vec{n}_j) \mathbb{1}_{\{v(h,|\vec{y}|+1)=j\}}
\end{aligned}$$

where $e_K = \sum_{x=1}^{K} e_{v(h,x),z(x)}$.

Let's give some explanations about the different terms of the equation. In the right hand side of the equation, the first term corresponds to the arrival of a customer from the outside. The second and third terms correspond to a service completion of a customer of class k in station i and a departure of this customer to the outside in the case of the second term and to another queue j as a customer of class m in the third term.

In the fourth term, after a completion of its service in station i, the customer of class k becomes a signal of type h which visits L_h service centers and succeeds with a certain probability in removing a customer at each visited station. Then it succeeds with probability $\alpha_{j,m}$ in putting the resulting customer in queue j as a class m customer. This is the case where the list length L_h is finite.

In the fifth term, we consider also finite lists, but in this case the signal does not succeed in putting the resulting customer in queue j.

The sixth and last term describes the case where the signal fails in removing a customer from a visited queue. There are 2 reasons for that. The first one corresponds to the case where the signal arrives to an empty queue and the second one to the case where it fails in the removal of the customer. In both cases the removal process stops and the signal vanishes. In this term, we also consider the case of infinite lists.

For the sake of readability, the proof of the theorem is in appendix A.

□

Now, we can compute by some trivial algebraic manipulations the steady-state distribution of the number of customers of each class in each queue. Let $\vec{u_i}$ be the vector whose components are u_i^k, the number of customers of class k in station i. Let $\vec{u}$ be the vector of vectors $\vec{u_i}$.

Theorem 3 *If system (8) has a solution, then the steady-state distribution $\pi(\vec{n})$ is given by:*

$$\pi(\vec{u}) = \prod_{i=1}^{N} h_i(\vec{u_i})$$

where the marginal probabilities $h_i(\vec{n}_i)$ have the following form:

$$h_i(\vec{u_i}) = \left(1 - \sum_{k=1}^{C}\right) |\vec{u_i}|! \prod_{k=1}^{C} \frac{(\rho_{i,k})^{u_{i,k}}}{u_{i,k}!}$$

We omit the proof of this theorem since the result is quite usual for multiple-class queues.

5 Examples, Related Works and Stability

Let's turn now into some details and examples about the model we presented in section 3. First, we present two examples to show some dynamics which may be described by multiple-class list oriented deletions. We also show that using a migration of signals described by a list rather than by a product of routing matrices may be more powerful to design models. This is already known in the context of the routing of customers using Kelly's formalism for customers classes.

We then explain how to obtain the main results on multiclass G-networks previously published using some restricted sets of lists. This, to the best of our knowledge, proves that our theorem generalizes most of the results on multiclass G-networks.

Finally, we present some results about the existence of a solution to traffic equation (8). Unlike in Jackson's network, the traffic equation of a G-network is not linear. Thus, the matrix P^+ properties are not sufficient to prove the existence of a solution to this equation. Usually, Brouwer's theorem [9] is used and as the traffic equation changes slightly for the various models, variants of the initial proof have been published [5][6][11][20]. We do not present here a new version of this proof. Instead, we investigate the existence of a solution to system (8) using a new type of arguments based on the equivalence between networks. Indeed, we show using the examples introduced in section 2 how the existence of a solution to traffic equation of a G-network with triggers implies the existence of a solution of a G-network with list oriented deletions. Furthermore, we show that these solutions are somehow related.

5.1 Examples

5.1.1 Specific signal routes

Consider the topology of the queueing network illustrated in figure 3. In this network, signals may be only of two types h_1 and h_2, and the customers migrating between stations are of 2 classes c_1 and c_2.

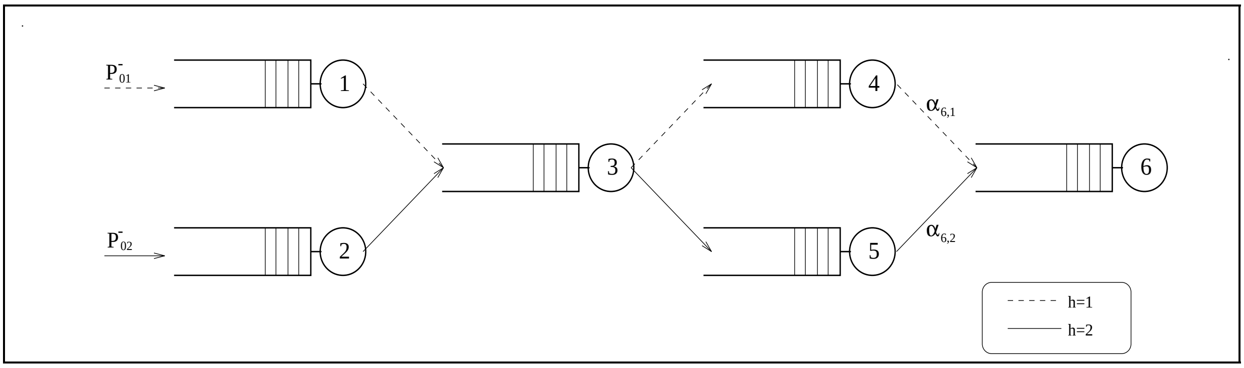

Figure 3: Routes of signals of types h_1 and h_2

Signals of type h_1 can enter the network only via station 1 with an arrival rate $\lambda_1^- = \frac{1}{2}$. To each signal of that type we associate the ordered list $S_{h_1} = \{1, 3, 4\}$. So, when a signal of type h_1 arrives into the network, it will try to remove a customer from stations 1, 3 and 4, in that order. If it succeeds, we assume that the signal will add a customer of class c_1 to station 6 with probability $\alpha_{6,c_1} = 1$. Customers which enter station 6 will leave the network just after their service completion.

Similarly, signals of type h_2 can enter the network only via station 2 with an arrival rate $\lambda_2^- = \frac{1}{2}$. To each signal of that type we associate the ordered list $S_{h_2} = \{2, 3, 5\}$. At the end of the deleting process, we assume that the signal of type h_2 will add a customer of class c_2 to station 6 with probability $\alpha_{6,c_2} = 1$.

We assume that the removing probabilities will depend only on the customer class. So, for all $x \in \mathbb{N}$, for all i ($i = 1, .., 5$), and for all h ($h = h_1, h_2$) we set $D_{i,c_1,h,x} = 1$ and $D_{i,c_2,h,x} = 0$. Furthermore, we define P_{0i} as the routing probability from outside the network to station i and we set $P^-_{01} = P^-_{02} = \frac{1}{2}$.

Assume now that the system is in steady-state and that state is $\vec{n} = (\vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_4, \vec{n}_5, \vec{n}_6)$.

For **Processor Sharing** discipline, each component of the system state vector $\vec{n}$ is noted $\vec{n}_i = (n_i^{c_1}, n_i^{c_2})$ where n_i^k is the number of customers of class k in station i. In this example, we assume that $\vec{n} = ((2,1), (1,2), (2,2), (3,0), (2,1), (0,0))$. The external arrivals of signals will lead to the states and according to the probabilities given by the following table:

Accessible state: $(\vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_4, \vec{n}_5, \vec{n}_6)$	Probability
$((2,1), (1,2), (2,2), (3,0), (2,1), (0,0))$	$\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{2}$
$((1,1), (1,2), (2,2), (3,0), (2,1), (0,0))$	$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{6}$
$((1,1), (1,2), (1,2), (2,0), (2,1), (1,0))$	$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{6}$
$((2,1), (0,2), (2,2), (3,0), (2,1), (0,0))$	$\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{12}$
$((2,1), (0,2), (1,2), (3,0), (2,1), (0,0))$	$\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{36}$
$((2,1), (0,2), (1,2), (3,0), (1,1), (0,1))$	$\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{18}$

For **LIFO/PR discipline**, the number of customers of each class in each station i is the same as for processor sharing discipline. However, the state vector $\vec{n}_i$ will be a vector which component $r_{i,x}$ denotes the class of the xth customer in the queue i. In this example, vectors $\vec{n}_i$ are defined as follows:

number of customers	state $\vec{n}_i = (r_{i,1}, r_{i,2}, \cdots)$
$\lvert\vec{n}_1\rvert = 3$	$\vec{n}_1 = (c_1, c_2, c_1)$
$\lvert\vec{n}_2\rvert = 3$	$\vec{n}_2 = (c_1, c_2, c_2)$
$\lvert\vec{n}_3\rvert = 4$	$\vec{n}_3 = (c_1, c_1, c_2, c_2)$
$\lvert\vec{n}_4\rvert = 3$	$\vec{n}_4 = (c_1, c_1, c_1)$
$\lvert\vec{n}_5\rvert = 3$	$\vec{n}_5 = (c_2, c_1, c_1)$
$\lvert\vec{n}_6\rvert = 0$	$\vec{n}_6 = ()$

The following table gives the number of possible states after the arrival of a signal into the network, with in each case the associated probability:

Accessible state $(\vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_4, \vec{n}_5, \vec{n}_6)$	Probability
$((c_2, c_1), (c_1, c_2, c_2), (c_1, c_2, c_2), (c_1, c_1), (c_2, c_1, c_1), (c_1))$	$\frac{1}{2}.1.1.1 = \frac{1}{2}$
$((c_1, c_2, c_1), (c_2, c_2), (c_1, c_2, c_2), (c_1, c_1, c_1), (c_2, c_1, c_1), ())$	$\frac{1}{2}.1.1 = \frac{1}{2}$

The results are presented as a product of fractions to emphasis elementary events of each transition.

5.1.2 Aperiodic cyclic trajectory of signals

In this example, we consider a network of two stations with 2 classes of customers and one type of signals (C=2 and H=1). Let c_1 and c_2 be the customer classes and h_1 the signal type.

We assume that external signals enter the network via station 2 with rate $\lambda_2^- = 1$. Therefore routing probabilities are $P_{01}^- = 0$ and $P_{02}^- = 1$. When a signal enters station 2, it will try to remove one customer from station 2, one customer from station 1, two customers from station 2, one customer from station 1, three customers from station 2, etc. So, this trajectory builds up the infinite list $S_{h_1} = \{\underbrace{2}_{1}, 1, \underbrace{2, 2}_{2}, 1, \underbrace{2, 2, 2}_{3}, 1, \underbrace{2, 2, 2, 2}_{4}, 1 \cdots\}$.

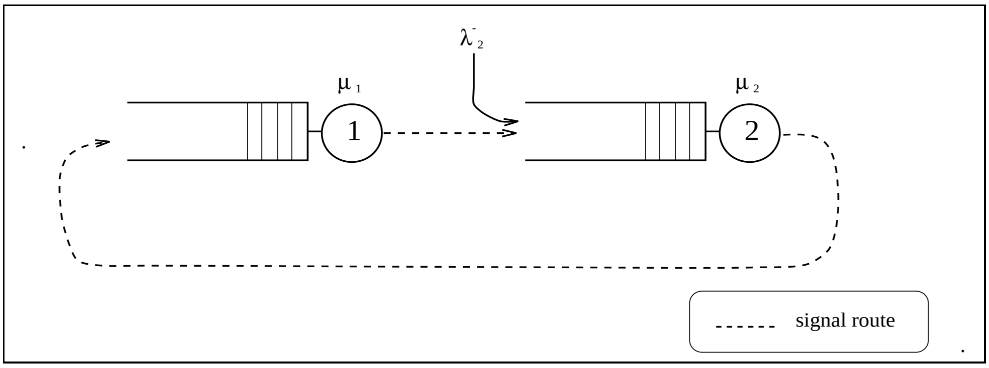

Figure 4: Aperiodic cycle trajectory

We assume that the service discipline in station 1 is Processor Sharing and the service discipline in station 2 is LIFO/PR. Furthermore, we consider that, for the processor sharing station (station 1), the deletion probability of a customer depends only on its class. A signal will succeed the removal of a customer of class c_1 with probability $D_{1,c_1,h_1,x} = 0$. In revenge, it will remove a customer of class c_2 with probability $D_{1,c_2,h_1,x} = 1$.

In station 2 where the service discipline is LIFO/PR, the deletion probability of a customer depends on the customer class and its position in the queue. So, for $x \in \mathbb{N}^\star$, a signal entering this queue will delete a customer of class c_1 with probability $D_{2,c_1,h_1,x} = \frac{1}{x}$ and a class c_2 customer with probability $D_{2,c_2,h_1,x} = 1 - \frac{1}{x}$.

We consider that the initial state of the network is $\vec{n} = (\vec{n}_1, \vec{n}_2)$ where $\vec{n}_1 = (n_1^{c1}, n_1^{c2}) = (1, 2)$ and $\vec{n}_2 = (r_{2,1}, r_{2,2}, r_{2,3}, r_{2,4}, r_{2,5}) = (c_1, c_2, c_1, c_2, c_1)$. The arrival of a signal into the network triggers a transition state according to the following probability distribution:

Accessible state: $(\vec{n}_1, \vec{n}_2)$	Probability
$((1,2),(c_1,c_2,c_1,c_2,c_1))$	$\left(1-\frac{1}{1}\right) = 0$
$((1,2),(c_2,c_1,c_2,c_1))$	$\frac{1}{1} \cdot \frac{1}{3} = \frac{1}{3}$
$((1,1),(c_2,c_1,c_2,c_1))$	$\frac{1}{1} \cdot \frac{2}{3}\left(\frac{1}{2}\right) = \frac{1}{3}$
$((1,1),(c_1,c_2,c_1))$	$\frac{1}{1} \cdot \frac{2}{3}\left(1-\frac{1}{2}\right) \cdot \left(\frac{2}{3}\right) = \frac{2}{9}$
$((1,1),(c_2,c_1))$	$\frac{1}{1} \cdot \frac{2}{3}\left(1-\frac{1}{2}\right) \cdot \frac{1}{3}\left(\frac{1}{2}\right) = \frac{1}{18}$
$((0,1),(c_2,c_1))$	$\frac{1}{1} \cdot \frac{2}{3}\left(1-\frac{1}{2}\right) \cdot \frac{1}{3} \cdot \frac{1}{2}\left(\frac{1}{4}\right) = \frac{1}{72}$
$((0,1),(c_1))$	$\frac{1}{1} \cdot \frac{2}{3}\left(1-\frac{1}{2}\right) \cdot \frac{1}{3} \cdot \frac{1}{2}\left(1-\frac{1}{4}\right) \cdot \left(1-\frac{1}{5}\right) = \frac{1}{30}$
$((0,1),())$	$\frac{1}{1} \cdot \frac{2}{3}\left(1-\frac{1}{2}\right) \cdot \frac{1}{3} \cdot \frac{1}{2}\left(1-\frac{1}{4}\right) \cdot \frac{1}{5} = \frac{1}{120}$

In this example, when a signal enters station 2, if the last customer which entered the station is of class c_1 then it succeeds the removal of at least one customer.

5.2 Related Works

In [21], Serfozo et al. present network processes where the transition rates are a product of a system-dependent transition rate and a string generation rate. These models include systems where the state of a station can be negative. A string is composed of vectors with negative or non negative entries. This result emphasizes the form of functional decomposition of service rates and transition initiation rates.

In this paper, we focus our study on Markovian networks that involve a non-negative state space ($\mathbb{N}^N$). We study the station nary behavior of G-networks in which lists are strings of the unit vectors $\vec{e_i}$ using Serfozo's formalism. Vector $\vec{e_i}$ represents one unit in station i. As the states of the systems are in $\mathbb{N}^N$, a transition involves a service completion of a batch of customers at various stations and the addition of one customer at another station. Our main theorem states that these G-networks have a product form solution with a traffic equation which depends on the list associated with the signal type.

The results we present in this paper are general since we consider G-networks with multiple classes of both customers and signals. Furthermore, and unlike Serfozo et al., we consider several service disciplines (PS, LIFO/PR and FIFO) for which we express and set clearly the dependence functions that are sufficient to preserve the product form solution. In this study, we also show precisely the trajectory of the effect of a signal in a transition.

To the best of our knowledge, almost all previously published results on multiclass G-networks may be formulated in terms of lists. Our formalism may also be used for models where only a single class of customers and a single class of signals are considered. Here below are some examples of the main results in which the list formalism may be used.

- The first result of Gelenbe on generalized networks [10] concerns networks with single class of customers and single class of signals. A signal has an effect on only one station, which in our model corresponds to the case where $L_h = 1$. Furthermore, if a signal succeeds the removal of one customer, then this customer is always triggered outside the network. So, $v(h, L_h + 1) = 0$. In relation with routing matrix notations, we have $P^-_{ij} = Q^h_{i,j} 1\!\!1_{\{v(h,1)=j\}}$ and $D_{v(h,x),z(x),h,x} = 1$ for all signal type h and all customer position x in the station $v(h, x)$.
- In [15], Gelenbe extends its first result to include the effect of batch removal. Following the notations introduced in this paper, the length of a list L_h is always finite and contains only one station index, say j, such that $S_h = \{j, j, \ldots, j\}$. Since in Gelenbe's model the batch of customers is always triggered outside, we then have for all h, $v(h, L_h + 1) = 0$. Since batches may be of different sizes and different contents, the number of signal types H can be infinite. As each customer batch corresponds to one signal list, we have then $P^-_{ij}\pi_{jl} = Q^h_{i,j} \prod_{x=1}^{l} 1\!\!1_{\{v(h,x)=j\}}$.
- In [14], Gelenbe presents another extension of its previous results called G-networks with triggered customer movements. The length of each list is reduced to one station ($L_h = 1$). In this model, a signal can trigger a customer from station j to station l with probability Q_{jl}. So, using the routing matrix notation, we have $P^-_{ij} Q_{jl} = Q^h_{i,j} 1\!\!1_{\{v(h,1)=j\}} 1\!\!1_{\{v(h,2)=l\}}$.
- In [3, 5], authors present a new version of G-networks with catastrophes. In these networks, a catastrophe is represented by a signal. If a signal enters a station, all customers in this station are immediately triggered outside the system. The networks considered in [5] are a multiple class version of G-networks with jumps back to zero. Using the list formalism, we can associate to each signal h an infinite list where then $L_h = \infty$ and $S_h = \{j, j, j, \cdots\}$. Moreover, we have, for $1 \leq i \leq N$, $P^{-(k)}_{ij} = Q^{k,h}_{i,j} 1\!\!1_{\{v(h,x)=j,\ x\in\mathbb{N}\}}$ and $D_{j,m,h,x} = 1$ for all station j, customer class m, signal type h and customer position x in the queue.
- In [4], monoclass G-networks of queues with batch services and customer coalescence are considered. Each station of this network is able to serve a batch of K_j customers. At a service completion at station j, if all customers of the batch are served, then these customers coalesce to form a single customer which then goes to another station l or leaves the network ($l = 0$). The associated list is such that $L_h = K_j$ and $v(h, L_h{+}1) = l$. Thus $S_h = \{j, j, \ldots, j\}$. The routing matrix notations are such that $P_{ij} = Q^h_{i,j} \prod_{x=1}^{K_j} 1\!\!1_{\{v(h,x)=j\}} 1\!\!1_{\{v(h,L_h+1)=l\}}$
- In [8], multiclass G-networks with cyclic synchronizations are presented. A cycle is defined on disconnected subsets of stations of the network. When a signal enters a cycle, it iterates the deletion of customers in each station of the cycle until it enters an empty station. Using our

formalism, the number of cycles will correspond to H and for each signal type h $L_h = +\infty$. For a cycle a with length L_a, the associated list is then $S_h = \{\underbrace{i_1, \ldots, i_L}_{L_a}, i_1, \ldots, i_L, \ldots\}$ where L_a is the number of stations involved in cycle a.

- In [6], multiclass G-networks with iterated deletions are studied. In these networks where several service disciplines are considered, when a signal enters a station an iterated deletion process starts. At each step of the deletion process, a customer is removed according to a probability which can depend on the service station and the class of the customer. The process stops when the deletion fails or the station is empty. Using the list formalism it is possible to represent this kind of dynamic by setting for all h, $L_h = +\infty$, and the list is $S_h = \{j, j, j, \cdots\}$. The probability of deletion depends on station j and customer class m only, $D_{j,m,h,x} = p_j^m$ and the routing matrix is restricted to $P_{ij}^{-(k)} = Q_{i,j}^{k,h} \mathbb{1}_{\{v(h,x)=j,\ x \in \mathbb{N}\}}$.
- In [22], networks with general signal routes correspond to monoclass version of our result with the restriction $D_{i,k,h,x} = p_i = 1$ for all station i. The number of signal types can be infinite. As in this paper, all signal routes are represented by a list of visited stations.

5.3 Existence of a solution to flow equation

Let us consider again the two examples introduced in section II. Again, we just present an example and for the sake of readability we only consider single class networks. The reduction argument that we use in the following applies also to multiclass networks.

Consider the G-network with triggers depicted in figure 1. Taking into account the arrival and service rates and the transition matrices, we obtain the following traffic equations:

$$\left\{ \begin{array}{lcl} \rho_1 & = & \frac{\lambda_1}{\mu_1} \\ \rho_2 & = & \frac{\lambda_2 + \sum_{i=1}^{3} P_{i2}^{+} \mu_i \rho_i}{\mu_2 + P_{12}^{-} \mu_1 \rho_1 + \mu_5 \rho_5} \\ \rho_3 & = & \frac{\lambda_3 + \sum_{i=1}^{3} P_{i3}^{+} \mu_i \rho_i}{\mu_3 + \mu_4 \rho_4} \\ \rho_4 & = & \frac{(P_{12}^{-} \mu_1 \rho_1 + \mu_5 \rho_5) \rho_2}{\mu_4} \\ \rho_5 & = & \frac{\mu_4 \rho_4 \rho_3}{\mu_5} \end{array} \right. \tag{10}$$

While the G-network with synchronized cyclic departures depicted in figure 2 is associated with the following traffic equations:

$$\left\{ \begin{array}{lcl} q_1 & = & \frac{\lambda_1}{\mu_1} \\ q_2 & = & \frac{\lambda_2 + \sum_{i=1}^{3} P_{i2}^{+} \mu_i q_i}{\mu_2 + \frac{P_{12}^{-} \mu_1 q_1}{1 - q_2 q_3}} \\ q_3 & = & \frac{\lambda_3 + \sum_{i=1}^{3} P_{i3}^{+} \mu_i q_i}{\mu_3 + \frac{P_{12}^{-} \mu_1 q_1 q_2}{1 - q_2 q_3}} \end{array} \right. \tag{11}$$

Assume that a solution exists to traffic equation (10). Then we can make the following substitution:

$$\mu_4 \rho_4 = (\mu_1 \rho_1 P_{12}^{-} + \mu_5 \rho_5) \rho_2 \quad \textit{and} \quad \mu_5 \rho_5 = \mu_4 \rho_4 \rho_3 \tag{12}$$

Therefore we get,

$$\mu_4\rho_4 = \frac{\mu_1\rho_1\rho_2 P^-_{12}}{1-\rho_2\rho_3} \quad and \quad \mu_5\rho_5 = \frac{\mu_1\rho_1\rho_2\rho_3 P^-_{12}}{1-\rho_2\rho_3} \tag{13}$$

Clearly, vector (ρ_1, ρ_2, ρ_3) is also solution of system (11). Therefore the existence of a solution to system (10) implies that a solution exists also for system (11).

Based on this result, we state the heuristic that, in all cases, using a G-network model with lists, we can build a model of G-networks with triggers with more service stations. Moreover, both models will have the same solution to their traffic equations. As the existence of a solution to G-networks with triggers have already been studied, it is then sufficient to consider this solution.

6 Conclusion

In this paper, we studied a new type of generalized networks of queues with steady-state product form solution. The dynamic of these networks is complex as we considered several classes of customers and signals, three types of service disciplines (FIFO,LIFO and PS) and a probabilistic deleting mechanism which is based on iterated deletions of customers according to an arbitrary list of stations. We showed that networks with a such dynamic has a product form solution and using new kind of arguments based on equivalence between networks we discussed the existence of a solution to the flow equation. This model is general since its formalism is, to the best of our knowledge, powerful enough to represent almost all previously published results on multiple class G-networks.

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A Technical Appendix

Before proceeding with the theorem proof, we need to introduce some technical lemmas.

Lemma 1 *: For any type of service center, the following relation holds:*

$$M_{i,k}(\vec{n}+e_{i,k})\frac{\Pi(\vec{n}+e_{i,k})}{\Pi(\vec{n})} = \mu_{i,k}\rho_{i,k}$$

Proof: The proof is purely algebraic.

Lemma 2 *:* $N_{i,k}(\vec{n}_i)$ *can be rewritten as* $N_{i,k}(\vec{n}_i) = 1 - \sum_{k=1}^{C} D_{i,k,h,x} f_{i,k}(\vec{n}_i)$ *where* $f_{i,k}(\vec{n}_i)$ *will depend on the service station type as follows:*

- *FIFO/LIFO:* $f_{i,k}(\vec{n}_i) = \mathbb{1}_{\{r_{i,1}=k\}}$
- *PS:* $f_{i,k}(\vec{n}_i) = \frac{n_i^k}{|\vec{n}_i|}\mathbb{1}_{\{|\vec{n}_i|>0\}}$

Proof: The proof consists of algebraic manipulations.

Lemma 3 *: The following relation holds for each service station type:*

$$Z(h,\vec{n},\vec{y})\frac{\Pi(\vec{n}+e_{|\vec{y}|})}{\Pi(\vec{n})} = \prod_{x=1}^{|\vec{y}|}\left(\rho_{v(h,x),z(x)}D_{v(h,x),z(x),h,x}\right)$$

Proof:

First, let's write the left hand side, noted F, of the equality using the definition of $e_{|\vec{y}|}$:

$$F = Z(h,\vec{n},\vec{y})\frac{\Pi(\vec{n}+\sum_{x=1}^{|\vec{y}|} e_{v(h,x),z(x)})}{\Pi(\vec{n})}$$

F can also be written as:

$$F = Z(h,\vec{n},\vec{y})\frac{\Pi\left(\vec{n}+\sum_{x=1}^{|\vec{y}|-1} e_{v(h,x),z(x)} + e_{v(h,|\vec{y}|),z(|\vec{y}|)}\right)}{\Pi(\vec{n})}$$

Now, the rest of the proof will depend on the service station type.

- FIFO/LIFO:

  As for such service station types $\frac{\Pi(\vec{n}+e_{i,k})}{\Pi(\vec{n})} = \rho_{i,k}$, we then have:

$$F = Z(h,\vec{n},\vec{y}) \frac{\Pi\left(\vec{n} + \sum_{x=1}^{|\vec{y}|-1} e_{v(h,x),z(x)}\right)}{\Pi(\vec{n})} \rho_{v(h,|\vec{y}|),z(|\vec{y}|)}$$

  By recurrence on $|\vec{y}|$, we then have:

$$F = Z(h,\vec{n},\vec{y}) \prod_{x=1}^{|\vec{y}|} \rho_{v(h,x),z(x)}$$

  Since for both service station types $Z(h,\vec{y},\vec{n}) = \prod_{x=1}^{|\vec{y}|} D_{v(h,x),z(x),h,x}$ we obtain:

$$F = \prod_{x=1}^{|\vec{y}|} \left(\rho_{v(h,x),z(x)} D_{v(h,x),z(x),h,x}\right)$$

  This completes the proof.

□

- PS:

  As for such service station type $\frac{\Pi(\vec{n}+e_{i,k})}{\Pi(\vec{n})} = \rho_{i,k} \frac{|n_i|+1}{n_i^k+1}$ we then have:

$$F = Z(h,\vec{n},\vec{y}) \frac{\Pi\left(\vec{n} + \sum_{x=1}^{|\vec{y}|-1} e_{v(h,x),z(x)}\right)}{\Pi(\vec{n})} \rho_{v(h,|\vec{y}|),z(|\vec{y}|)} \frac{|n_{v(h,|\vec{y}|)}|+1}{n_{v(h,|\vec{y}|)}^{z(|\vec{y}|)}+1}$$

  By recurrence on $|\vec{y}|$, we then have:

$$F = Z(h,\vec{n},\vec{y}) \left(\rho_{v(h,|\vec{y}|),z(|\vec{y}|)} \frac{|n_{v(h,|\vec{y}|)}|+1}{n_{v(h,|\vec{y}|)}^{z(|\vec{y}|)}+1}\right) \left(\rho_{v(h,|\vec{y}|-1),z(|\vec{y}|-1)} \frac{|n_{v(h,|\vec{y}|-1)}|+1}{n_{v(h,|\vec{y}|-1)}^{z(|\vec{y}|-1)}+1}\right) \cdots \left(\rho_{v(h,1),z(1)} \frac{|n_{v(h,1)}|+1}{n_{v(h,1)}^{z(1)}+1}\right)$$

  Replacing Z by its expression for PS service center, we get:

$$F = \prod_{x=1}^{|\vec{y}|} \left(\rho_{v(h,x),z(x)} D_{v(h,x),z(x),h,x}\right)$$

  This completes the proof.

□

Proof of the theorem 2

Let us now prove that the solution given by the theorem satisfies the Chapmann-Kolmogorov equation.

$$\begin{aligned}
\Pi(\vec{n}) \sum_{i=1}^{N}\sum_{k=1}^{C} \left[\lambda_i^{(k)} + M_{i,k}(\vec{n}_i)\right] = \\
& \sum_{i=1}^{N}\sum_{k=1}^{C} \lambda_i^{(k)} A_{i,k}(\vec{n}_i)\Pi(\vec{n}-e_{i,k}) \\
+ & \sum_{i=1}^{N}\sum_{k=1}^{C} M_{i,k}(\vec{n}+e_{i,k}) d_i^{(k)} \Pi(\vec{n}+e_{i,k}) \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{C}\sum_{m=1}^{C} M_{i,k}(\vec{n}+e_{i,k}-e_{j,m}) A_{j,m}(\vec{n}_j) P_{ij}^{+(k,m)} \Pi(\vec{n}+e_{i,k}-e_{j,m}) \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{C}\sum_{m=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Gamma(h)} M_{i,k}(\vec{n}+e_{i,k}+e_{L_h}-e_{j,m}) A_{j,m}(\vec{n}_j) Q_{i,j}^{k,h} \Pi(\vec{n}+e_{i,k}+e_{L_h}-e_{j,m}) \\
& \qquad Z(h,\vec{n},\vec{y})\alpha_{j,m} 1\!\!1_{\{v(h,L_h+1)=j\}} 1\!\!1_{\{L_h<\infty\}} \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{C}\sum_{m=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Gamma(h)} M_{i,k}(\vec{n}+e_{i,k}+e_{L_h}) Q_{i,j}^{k,h} \Pi(\vec{n}+e_{i,k}+e_{L_h}) \\
& \qquad Z(h,\vec{n},\vec{y})(1-\alpha_{j,m}) 1\!\!1_{\{v(h,L_h+1)=j\}} 1\!\!1_{\{L_h<\infty\}} \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{C}\sum_{m=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Lambda(h)} M_{i,k}(\vec{n}+e_{i,k}+e_{|\vec{y}|}) Q_{i,j}^{k,h} \Pi(\vec{n}+e_{i,k}+e_{|\vec{y}|}) \\
& \qquad Z(h,\vec{n},\vec{y}) N_{j,m}(\vec{n}_j) 1\!\!1_{\{v(h,|\vec{y}|+1)=j\}}
\end{aligned}$$

Dividing both sides of the equation by $\Pi(\vec{n})$ and using lemmas 1 and 3, we have:

$$
\begin{aligned}
&\sum_{i=1}^{N}\sum_{k=1}^{C}\left[\lambda_i^{(k)}+M_{i,k}(\vec{n}_i)\right]= \\
&\qquad \sum_{i=1}^{N}\sum_{k=1}^{C}\lambda_i^{(k)}A_{i,k}(\vec{n}_i)\frac{\Pi(\vec{n}-e_{i,k})}{\Pi(\vec{n})} \\
&\quad + \sum_{i=1}^{N}\sum_{k=1}^{C}\mu_{i,k}\rho_{i,k}d_i^{(k)} \\
&\quad + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{C}\sum_{m=1}^{C}\mu_{i,k}\rho_{i,k}P_{ij}^{+(k,m)}A_{j,m}(\vec{n}_j)\frac{\Pi(\vec{n}-e_{j,m})}{\Pi(\vec{n})} \\
&\quad + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{C}\sum_{m=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Gamma(h)}\mu_{i,k}\rho_{i,k}Q_{i,j}^{k,h}\prod_{x=1}^{L_h}\left(\rho_{v(h,x),z(x)}D_{v(h,x),z(x),h,x}\right)\frac{\Pi(\vec{n}-e_{j,m})}{\Pi(\vec{n})} \\
&\qquad\qquad A_{j,m}(\vec{n}_j)\alpha_{j,m}\mathbb{1}_{\{v(h,L_h+1)=j\}}\mathbb{1}_{\{L_h<\infty\}} \\
&\quad + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{C}\sum_{m=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Gamma(h)}\mu_{i,k}\rho_{i,k}Q_{i,j}^{k,h}\prod_{x=1}^{L_h}\left(\rho_{v(h,x),z(x)}D_{v(h,x),z(x),h,x}\right)(1-\alpha_{j,m}) \\
&\qquad\qquad \mathbb{1}_{\{v(h,L_h+1)=j\}}\mathbb{1}_{\{L_h<\infty\}} \\
&\quad + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Lambda(h)}\mu_{i,k}\rho_{i,k}Q_{i,j}^{k,h}\prod_{x=1}^{|\vec{y}|}\left(\rho_{v(h,x),z(x)}D_{v(h,x),z(x),h,x}\right)N_{j,m}(\vec{n}_j)\mathbb{1}_{\{v(h,|\vec{y}|+1)=j\}}
\end{aligned}
$$

Let's now define $\Lambda_{i,k}^{+}$ and $\Lambda_{i,k}^{-}$ as follows:

$$
\begin{cases}
\Lambda_{i,k}^{+} = \displaystyle\sum_{j=1}^{N}\sum_{h=1}^{H}\sum_{\vec{y}\in\Gamma(h)}\mu_{j,m}\rho_{j,m}Q_{j,i}^{m,h}\prod_{x=1}^{L_h}\left(\rho_{v(h,x),z(x)}D_{v(h,x),z(x),h,x}\right)\alpha_{i,k}\mathbb{1}_{\{v(h,L_h+1)=i\}}\mathbb{1}_{\{L_h<\infty\}} \\
\Lambda_{i,k}^{-} = \displaystyle\sum_{j=1}^{N}\sum_{m=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Lambda(h)}\mu_{j,m}\rho_{j,m}Q_{j,i}^{m,h}\prod_{x=1}^{|\vec{y}|}\left(\rho_{v(h,x),z(x)}D_{v(h,x),z(x),h,x}\right)D_{i,k,h,|\vec{y}|+1}\mathbb{1}_{\{v(h,|\vec{y}|+1)=i\}}
\end{cases}
$$

Grouping the first term with the third and the fourth terms, we obtain:

$$
\begin{aligned}
&\sum_{i=1}^{N}\sum_{k=1}^{C}\left[\lambda_i^{(k)}+M_{i,k}(\vec{n}_i)\right]= \\
&\qquad \sum_{i=1}^{N}\sum_{k=1}^{C}\left(\lambda_i^{(k)}+\sum_{j=1}^{N}\sum_{m=1}^{C}\mu_{j,m}\rho_{j,m}P_{ji}^{+(m,k)}+\Lambda_{i,k}^{+}\right)A_{i,k}(\vec{n}_i)\frac{\Pi(\vec{n}-e_{i,k})}{\Pi(\vec{n})} \\
&\quad + \sum_{i=1}^{N}\sum_{k=1}^{C}\mu_{i,k}\rho_{i,k}d_i^{(k)} \\
&\quad + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{C}\sum_{m=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Gamma(h)}\mu_{i,k}\rho_{i,k}Q_{i,j}^{k,h}\prod_{x=1}^{L_h}\left(\rho_{v(h,x),z(x)}D_{v(h,x),z(x),h,x}\right)(1-\alpha_{j,m}) \\
&\qquad\qquad \mathbb{1}_{\{v(h,L_h+1)=j\}}\mathbb{1}_{\{L_h<\infty\}} \\
&\quad + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Lambda(h)}\mu_{i,k}\rho_{i,k}Q_{i,j}^{k,h}\prod_{x=1}^{|\vec{y}|}\left(\rho_{v(h,x),z(x)}D_{v(h,x),z(x),h,x}\right)N_{j,m}(\vec{n}_j)\mathbb{1}_{\{v(h,|\vec{y}|+1)=j\}}
\end{aligned}
$$

Using lemma (2), we then have:

$$
\begin{aligned}
&\sum_{i=1}^{N}\sum_{k=1}^{C}\left[\lambda_i^{(k)} + M_{i,k}(\vec{n}_i)\right] = \\
&\quad \sum_{i=1}^{N}\sum_{k=1}^{C}\left(\lambda_i^{(k)} + \sum_{j=1}^{N}\sum_{m=1}^{C}\mu_{j,m}\rho_{j,m}P_{ji}^{+(m,k)} + \Lambda_{i,k}^{+}\right) A_{i,k}(\vec{n}_i)\frac{\Pi(\vec{n}-e_{i,k})}{\Pi(\vec{n})} \\
&+ \sum_{i=1}^{N}\sum_{k=1}^{C}\mu_{i,k}\rho_{i,k}d_i^{(k)} \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{C}\sum_{m=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Gamma(h)}\mu_{i,k}\rho_{i,k}Q_{i,j}^{k,h}\prod_{x=1}^{L_h}\left(\rho_{v(h,x),z(x)}D_{v(h,x),z(x),h,x}\right)(1-\alpha_{j,m})\mathbb{1}_{\{v(h,L_h+1)=j\}}\mathbb{1}_{\{L_h<\infty\}} \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Lambda(h)}\mu_{i,k}\rho_{i,k}Q_{i,j}^{k,h}\prod_{x=1}^{|\vec{y}|}\left(\rho_{v(h,x),z(x)}D_{v(h,x),z(x),h,x}\right)\left(1-\sum_{m=1}^{C}D_{j,m,h,|\vec{y}|+1}f_{j,m}(\vec{n}_j)\right) \\
&\qquad \mathbb{1}_{\{v(h,|\vec{y}|+1)=j\}}
\end{aligned}
$$

Using the definition of $\Lambda_{i,k}^{-}$ and putting the negative part of the right hand side in the left hand side of the previous equation, we obtain:

$$
\begin{aligned}
&\sum_{i=1}^{N}\sum_{k=1}^{C}\lambda_i^{(k)} + \sum_{i=1}^{N}\sum_{k=1}^{C}\left[M_{i,k}(\vec{n}_i) + \Lambda_{i,k}^{-}f_{i,k}(\vec{n}_i)\right] = \\
&\quad \sum_{i=1}^{N}\sum_{k=1}^{C}\left(\lambda_i^{(k)} + \sum_{j=1}^{N}\sum_{m=1}^{C}\mu_{j,m}\rho_{j,m}P_{ji}^{+(m,k)} + \Lambda_{i,k}^{+}\right) A_{i,k}(\vec{n}_i)\frac{\Pi(\vec{n}-e_{i,k})}{\Pi(\vec{n})} \\
&+ \sum_{i=1}^{N}\sum_{k=1}^{C}\mu_{i,k}\rho_{i,k}d_i^{(k)} \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{C}\sum_{m=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Gamma(h)}\mu_{i,k}\rho_{i,k}Q_{i,j}^{k,h}\prod_{x=1}^{L_h}\left(\rho_{v(h,x),z(x)}D_{v(h,x),z(x),h,x}\right)(1-\alpha_{j,m}) \\
&\qquad \mathbb{1}_{\{v(h,L_h+1)=j\}}\mathbb{1}_{\{L_h<\infty\}} \\
&+ \sum_{i=1}^{N}\sum_{k=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Lambda(h)}\mu_{i,k}\rho_{i,k}Q_{i,j}^{k,h}\prod_{x=1}^{|\vec{y}|}\left(\rho_{v(h,x),z(x)}D_{v(h,x),z(x),h,x}\right)
\end{aligned}
$$

The previous system can be decomposed into 2 equations, a flow equation and a state-dependent equation as follows:

$$\sum_{i=1}^{N}\sum_{k=1}^{C}\lambda_i^{(k)} =$$

$$\begin{aligned}
& \sum_{i=1}^{N}\sum_{k=1}^{C}\mu_{i,k}\rho_{i,k}d_i^{(k)} \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{C}\sum_{m=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Gamma(h)}\mu_{i,k}\rho_{i,k}Q_{i,j}^{k,h}\prod_{x=1}^{L_h}\left(\rho_{v(h,x),z(x)}D_{v(h,x),z(x),h,x}\right)(1-\alpha_{j,m}) \\
& \qquad \mathbb{1}_{\{v(h,L_h+1)=j\}}\mathbb{1}_{\{L_h<\infty\}} \\
+ & \sum_{i=1}^{N}\sum_{k=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Lambda(h)}\mu_{i,k}\rho_{i,k}Q_{i,j}^{k,h}\prod_{x=1}^{|\vec{y}|}\left(\rho_{v(h,x),z(x)}D_{v(h,x),z(x),h,x}\right)
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
& \sum_{i=1}^{N}\sum_{k=1}^{C}M_{i,k}(\vec{n}_i)+\sum_{i=1}^{N}\sum_{k=1}^{C}\Lambda_{i,k}^{-}f_{i,k}(\vec{n}_i) = \\
& \quad \sum_{i=1}^{N}\sum_{k=1}^{C}\left(\lambda_i^{(k)}+\sum_{j=1}^{N}\sum_{m=1}^{C}\mu_{j,m}\rho_{j,m}P_{ji}^{+(m,k)}+\Lambda_{i,k}^{+}\right)A_{i,k}(\vec{n}_i)\frac{\Pi(\vec{n}-e_{i,k})}{\Pi(\vec{n})}
\end{aligned} \tag{15}$$

Let's first prove that equation (14) is a flow equation. This equation can be rewritten as follows:

$$\sum_{i=1}^{N}\sum_{k=1}^{C}\lambda_i^{(k)} =$$

$$\begin{aligned}
& \sum_{i=1}^{N}\sum_{k=1}^{C}\mu_{i,k}\rho_{i,k}d_i^{(k)} \\
+ & \sum_{i=1}^{N}\sum_{k=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Gamma(h)}\mu_{i,k}\rho_{i,k}Q_{i,j}^{k,h}\prod_{x=1}^{L_h}\left(\rho_{v(h,x),z(x)}D_{v(h,x),z(x),h,x}\right)\mathbb{1}_{\{L_h<\infty\}} \\
- & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{C}\sum_{m=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Gamma(h)}\mu_{i,k}\rho_{i,k}Q_{i,j}^{k,h}\prod_{x=1}^{L_h}\left(\rho_{v(h,x),z(x)}D_{v(h,x),z(x),h,x}\right)\alpha_{j,m}\mathbb{1}_{\{v(h,L_h+1)=j\}}\mathbb{1}_{\{L_h<\infty\}} \\
+ & \sum_{i=1}^{N}\sum_{k=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Lambda(h)}\mu_{i,k}\rho_{i,k}Q_{i,j}^{k,h}\prod_{x=1}^{|\vec{y}|}\left(\rho_{v(h,x),z(x)}D_{v(h,x),z(x),h,x}\right)
\end{aligned}$$

Putting the negative term of the right hand side of the equation in its left hand side, we then obtain:

$$\sum_{i=1}^{N}\sum_{k=1}^{C}\lambda_i^{(k)} + \sum_{i=1}^{N}\sum_{k=1}^{C}\Lambda_{i,k}^{+} =$$
$$\sum_{i=1}^{N}\sum_{k=1}^{C}\mu_{i,k}\rho_{i,k}d_i^{(k)}$$
$$+ \sum_{i=1}^{N}\sum_{k=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Gamma(h)}\mu_{i,k}\rho_{i,k}Q_{i,j}^{k,h}\prod_{x=1}^{L_h}\left(\rho_{v(h,x),z(x)}D_{v(h,x),z(x),h,x}\right)\mathbb{1}_{\{L_h<\infty\}}$$
$$+ \sum_{i=1}^{N}\sum_{k=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Lambda(h)}\mu_{i,k}\rho_{i,k}Q_{i,j}^{k,h}\prod_{x=1}^{|\vec{y}|}\left(\rho_{v(h,x),z(x)}D_{v(h,x),z(x),h,x}\right)$$

The last term of this equation refers to 2 cases, the first one is the case where the list size is infinite ($L_h = \infty$) and the second one is the case where the signal h does not succeed in deleting all the customers in the list ($|\vec{y}| < L_h$). Considering these two options, last system can be rewritten as follows:

$$\sum_{i=1}^{N}\sum_{k=1}^{C}\lambda_i^{(k)} + \sum_{i=1}^{N}\sum_{k=1}^{C}\Lambda_{i,k}^{+} =$$
$$\sum_{i=1}^{N}\sum_{k=1}^{C}\mu_{i,k}\rho_{i,k}d_i^{(k)}$$
$$+ \sum_{i=1}^{N}\sum_{k=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Gamma(h)\cup\Lambda(h)}\mu_{i,k}\rho_{i,k}Q_{i,j}^{k,h}\prod_{x=1}^{|\vec{y}|}\left(\rho_{v(h,x),z(x)}D_{v(h,x),z(x),h,x}\right)\mathbb{1}_{\{L_h<\infty\}}$$
$$+ \sum_{i=1}^{N}\sum_{k=1}^{C}\sum_{h=1}^{H}\sum_{\vec{y}\in\Lambda(h)}\mu_{i,k}\rho_{i,k}Q_{i,j}^{k,h}\prod_{x=1}^{|\vec{y}|}\left(\rho_{v(h,x),z(x)}D_{v(h,x),z(x),h,x}\right)\mathbb{1}_{\{L_h=\infty\}}$$

Obviously this equation is a flow equation. This completes the proof.

□

Now, let's prove that equality (15) holds for each service station type.

- FIFO:

$$\sum_{k=1}^{C}\left(M_{i,k}(\vec{n}_i) + \Lambda_{i,k}^{-}f_{i,k}(\vec{n}_i)\right) = \sum_{k=1}^{C}\left(\lambda_i^{(k)} + \sum_{j=1}^{N}\sum_{m=1}^{C}\mu_{j,m}\rho_{j,m}P_{ji}^{+(m,k)} + \Lambda_{i,k}^{+}\right)A_{i,k}(\vec{n}_i)\frac{\Pi(\vec{n}-e_{i,k})}{\Pi(\vec{n})}$$

Using the definition of $M_{i,k}(\vec{n}_i)$ and $f_{i,k}(\vec{n}_i)$ for a FIFO station, we get:

$$\sum_{k=1}^{C}\left(\mu_{i,k} + \Lambda_{i,k}^{-}\right)\mathbb{1}_{\{r_{i,1}=k\}} = \sum_{k=1}^{C}\left(\lambda_i^{(k)} + \sum_{j=1}^{N}\sum_{m=1}^{C}\mu_{j,m}\rho_{j,m}P_{ji}^{+(m,k)} + \Lambda_{i,k}^{+}\right)\frac{\Pi(\vec{n}-e_{i,k})}{\Pi(\vec{n})}\mathbb{1}_{\{r_{i,\infty}=k\}}$$

After some algebraic manipulations and using the expression of $\rho_{i,k}$, we then get:

$$\sum_{k=1}^{C} \left(\mu_{i,k} + \Lambda^-_{i,k}\right) 1\!\!1_{\{r_{i,1}=k\}} = \sum_{k=1}^{C} \left(\mu_{i,k} + \Lambda^-_{i,k}\right) 1\!\!1_{\{r_{i,\infty}=k\}}$$

To be satisfied, the equality imposes that for all service station i, customer class k, signal type h and position x in the list:

$$\begin{cases} \mu_{i,k} & = & \mu_i \\ D_{i,k,h,x} & = & D_i \end{cases}$$

□

- LIFO:

$$\sum_{k=1}^{C} \left(M_{i,k}(\vec{n}_i) + \Lambda^-_{i,k} f_{i,k}(\vec{n}_i)\right) = \sum_{k=1}^{C} \left(\lambda_i^{(k)} + \sum_{j=1}^{N}\sum_{m=1}^{C} \mu_{j,m}\rho_{j,m} P_{ji}^{+(m,k)} + \Lambda^+_{i,k}\right) A_{i,k}(\vec{n}_i) \frac{\Pi(\vec{n} - e_{i,k})}{\Pi(\vec{n})}$$

Using the definition of $M_{i,k}(\vec{n}_i)$ and $f_{i,k}(\vec{n}_i)$ for a LIFO station, we get:

$$\sum_{k=1}^{C} \left(\mu_{i,k} + \Lambda^-_{i,k}\right) 1\!\!1_{\{r_{i,1}=k\}} = \sum_{k=1}^{C} \left(\lambda_i^{(k)} + \sum_{j=1}^{N}\sum_{m=1}^{C} \mu_{j,m}\rho_{j,m} P_{ji}^{+(m,k)} + \Lambda^+_{i,k}\right) \frac{\Pi(\vec{n} - e_{i,k})}{\Pi(\vec{n})} 1\!\!1_{\{r_{i,1}=k\}}$$

After some algebraic manipulations and using the expression of $\rho_{i,k}$, we then get:

$$\sum_{k=1}^{C} \left(\mu_{i,k} + \Lambda^-_{i,k}\right) 1\!\!1_{\{r_{i,1}=k\}} = \sum_{k=1}^{C} \mu_{i,k} 1\!\!1_{\{r_{i,1}=k\}} + \sum_{k=1}^{C} \Lambda^-_{i,k} 1\!\!1_{\{r_{i,1}=k\}}$$

This leads to the following equality:

$$\sum_{k=1}^{C} \left(\mu_{i,k} + \Lambda^-_{i,k}\right) 1\!\!1_{\{r_{i,1}=k\}} = \sum_{k=1}^{C} \left(\mu_{i,k} + \Lambda^-_{i,k}\right) 1\!\!1_{\{r_{i,1}=k\}}$$

This completes the proof.

□

- PS:

$$\sum_{k=1}^{C}\Big(M_{i,k}(\vec{n}_i)+\Lambda^-_{i,k}f_{i,k}(\vec{n}_i)\Big)=\sum_{k=1}^{C}\left(\lambda_i^{(k)}+\sum_{j=1}^{N}\sum_{m=1}^{C}\mu_{j,m}\rho_{j,m}P_{ji}^{+(m,k)}+\Lambda^+_{i,k}\right)A_{i,k}(\vec{n}_i)\frac{\Pi(\vec{n}-e_{i,k})}{\Pi(\vec{n})}$$

Using the definition of $M_{i,k}(\vec{n}_i)$ and $f_{i,k}(\vec{n}_i)$ for a PS station, we get:

$$\sum_{k=1}^{C}\Big(\mu_{i,k}+\Lambda^-_{i,k}\Big)\frac{n_i^k}{|\vec{n}_i|}1\!\!1_{\{|\vec{n}_i|>0\}}=\sum_{k=1}^{C}\left(\lambda_i^{(k)}+\sum_{j=1}^{N}\sum_{m=1}^{C}\mu_{j,m}\rho_{j,m}P_{ji}^{+(m,k)}+\Lambda^+_{i,k}\right)\frac{\Pi(\vec{n}-e_{i,k})}{\Pi(\vec{n})}1\!\!1_{\{n_i^k>0\}}$$

Using equation (8), we then have:

$$\sum_{k=1}^{C}\Big(\mu_{i,k}+\Lambda^-_{i,k}\Big)\frac{n_i^k}{|\vec{n}_i|}1\!\!1_{\{|\vec{n}_i|>0\}}=\sum_{k=1}^{C}\Big(\mu_{i,k}+\Lambda^-_{i,k}\Big)\frac{n_i^k}{|\vec{n}_i|}1\!\!1_{\{n_i^k>0\}}$$

Since $n_i^k>0$ implies that $|\vec{n}_i|>0$, this equation can then be rewritten as follows:

$$\sum_{k=1}^{C}\Big(\mu_{i,k}+\Lambda^-_{i,k}\Big)\frac{n_i^k}{|\vec{n}_i|}1\!\!1_{\{|\vec{n}_i|>0\}}=\sum_{k=1}^{C}\Big(\mu_{i,k}+\Lambda^-_{i,k}\Big)\frac{n_i^k}{|\vec{n}_i|}1\!\!1_{\{|n_i|>0\}}$$

This completes the proof.

□